SCHEME THEORETIC TROPICALIZATION

OLIVER LORSCHEID

ABSTRACT. In this paper, we introduce ordered blueprints and ordered blue schemes, which serve as a common language for the different approaches to tropicalizations and which enhances tropical varieties with a schematic structure. As an abstract concept, we consider a tropicalization as a moduli problem about extensions of a given valuation $v : k \to T$ between ordered blueprints k and T. If T is idempotent, then we show that a generalization of the Giansiracusa bend relation leads to a representing object for the tropicalization, and that it has yet another interpretation in terms of a base change along v. We call such a representing object a *scheme theoretic tropicalization*.

This theory recovers and improves other approaches to tropicalizations as we explain with care in the second part of this text.

The Berkovich analytification and the Kajiwara-Payne tropicalization appear as rational point sets of a scheme theoretic tropicalization. The same holds true for its generalization by Foster and Ranganathan to higher rank valuations.

The scheme theoretic Giansiracusa tropicalization can be recovered from the scheme theoretic tropicalizations in our sense. We obtain an improvement due to the resulting blueprint structure, which is sufficient to remember the Maclagan-Rincón weights.

The Macpherson analytification has an interpretation in terms of a scheme theoretic tropicalization, and we give an alternative approach to Macpherson's construction of tropicalizations.

The Thuillier analytification and the Ulirsch tropicalization are rational point sets of a scheme theoretic tropicalization. Our approach yields a generalization to any, possibly nontrivial, valuation $v : k \to T$ with idempotent T and enhances the tropicalization with a schematic structure.

CONTENTS

2000 *Mathematics Subject Classification.* Primary 14A20, 14G22, 14T05. Secondary 12J20, 13A18, 16Y60.

Introduction

The purpose of this paper is the development of a language that allows us to consider the different techniques of tropicalizing a classical scheme within the same framework and that provides a scheme theoretic structure for tropicalizations. In order to explain the relevance of our results, we begin with an outline of the historical development.

History. In spite of the early works of Bergman ([7]) and Bieri and Groves ([11]), tropical geometry became an active research area only 15 years ago when it became clear that the combinatorial nature of tropical varieties could be used to study their classical counterparts; for instance, see Mikhalkin's celebrated computation of Gromov-Witten invariants ([41]).

Let k be a field and X a closed subvariety of the torus $(k^\times)^n$. From its early days on, the tropicalization $\mathrm{Trop}(X)$ along a logarithmic valuation $v: k^\times \to \mathbb{R}$ was equally understood as an amoeba, as the corner locus of its defining polynomials and as the coordinatewise evaluation of seminorms extending v ([11], [17]). It was known that $\mathrm{Trop}(X)$ can be endowed with the structure of a finite polyhedral complex in $\mathbb{R}^n$, whose top dimensional polyhedra carry weights that satisfy a certain balancing condition ([11], [48]). It was also clear that the tropicalization of X could be compactified via an embedding of $(k^\times)^n$ into a toric variety ([40], [42], [48]).

However, it took some years till this knowledge found a clear formulation in the independent works of Kajiwara ([26]) and Payne ([45]) who defined the tropicalization of a closed subvariety X of a toric variety along a nonarchimedean valuation as a quotient of the Berkovich space of X, which can be understood as a stack quotient ([54]).

From this point on, the understanding of tropicalization was broadened in different directions. One important source of inspiration were skeleta of Berkovich spaces, as introduced by Berkovich ([9]). In the situation of a variety over a discretely valued field, a semistable model over the discrete valuation ring defines a skeleton for the Berkovich space. While a strict correspondence between skeleta from semistable models and tropicalizations holds only in special situations, a generalized framework of skeleta for semistable pairs illuminated this relation; cf. Tyomkin ([52]), Baker, Payne and Rabinoff ([4], [5]), and Gubler, Rabinoff and Werner ([24], [25]).

An important variant of skeleta for semistable models is Thuillier's theory of skeleta for toroidal embeddings over trivially valued fields ([50]). Abramovich, Caporaso and Payne interpreted these skeleta as tropicalizations ([1]), and Ulirsch ([53]) clarified this process in terms of a tropicalization associated to fine and saturated log schemes, which passes through an associated Kato fan and the local tropicalization of Popescu-Pampu and Stepanov ([47]). Ulirsch's tropicalization of fine and saturated Zariski log schemes coincides with the approach of Gross and Siebert ([23]) in their study of logarithmic Gromov-Witten invariants.

A recent variant of the Kajiwara-Payne tropicalization replaces the valuation $v: k^\times \to \mathbb{R}$ by a valuation $v: k^\times \to \mathbb{R}^n$ of higher rank. This was first considered by Banerjee ([6]) in the case of higher local fields, and the idea was taken up and generalized by Foster and Ranganathan ([19]), who showed that higher rank tropicalizations reflect certain properties of classical varieties over k.

With the progress of generalized scheme theory, often coined as $\mathbb{F}_1$-geometry, a theory of semiring schemes and, in particular, schemes over the tropical numbers became available; see the work of Durov ([16]), Toën and Vaquié ([51]), and the author ([32]). Jeff and Noah Giansiracusa used this theory in the case of closed subschemes of toric varieties to enhance the tropicalization with a schematic structure ([21]). At the same time, Macpherson endowed such a tropicalization with the structure of an analytic space ([38]). Strikingly, Maclagan and Rincón showed that the schematic structure of the tropicalization together with the embedding into an

ambient torus encodes the structure of the tropical variety as a balanced weighted polyhedral complex ([36]).

In the following, we will explain how to put these different approaches to tropicalizations on a common footing via ordered blueprints.

From coordinates to blueprints. Grosso modo, a tropicalization of a k-scheme X is the image of certain chosen coordinates of X under a valuation v of the field k. The coordinates for X can be given by different means: an embedding of X into affine space or into a toric variety; a simple normal crossing divisor on X; a simple toroidal embedding; a fine and saturated log structure for X.

For simplicity, let $X = \operatorname{Spec} R$ be an affine k-scheme. The choice of coordinates singles out a multiplicative subset A of R. In case of an closed immersion $\iota : X \to \operatorname{Spec} k[A_0]$ into a toric variety, A equals the set of elements of the form $\Gamma\iota(c \cdot a) \in R$ where $c \in k$ and $a \in A_0$ and $\Gamma\iota : k[A_0] \to R$ is the map between the respective global sections. In case of the complement $U \subset X$ of a simple normal crossing divisor, or, more general, a simple toroidal embedding $U \subset X$, the multiplicative set A equals the intersection $R \cap \mathcal{O}_X(U)^\times$. In case of a fine and saturated log structure $\alpha : \mathcal{M}_X \to \mathcal{O}_X$, the multiplicative subset is $A = \mathcal{M}_X(X)$.

Let B^+ be the subring of R that is generated by A. Then the inclusion $A \subset B^+$ is a blueprint in the sense of [32]. In this paper, we explain how to tropicalize X with respect to the choice of blueprint $B = (A \subset B^+)$, and we show that this recovers the previously mentioned concepts of tropicalization.

Analytification as a base change. Before we enter the theory of ordered blueprints, we want to explain the underlying idea that is inspired by Paugam's approach [44] to analytic geometry. The following is a simplified account of this theory.

An ordered semiring is a (commutative) semiring R (with 0 and 1) together a partial order $\leqslant$ that is additive and multiplicative. A *subadditive homomorphism of ordered semirings* is an order preserving multiplicative map $f : R_1 \to R_2$ with $f(0) = 0$, $f(1) = 1$ and $f(a+b) \leqslant f(a) + f(b)$.

This allows us to perform the following gedankenexperiments. We consider rings as trivially ordered semirings. Note that a subadditive homomorphism between trivially ordered semirings is always additive, which means that subadditive homomorphisms of rings are homomorphisms.

If we endow the semiring $\mathbb{R}_{\geqslant 0}$ with its natural total order, then a seminorm $v : k \to \mathbb{R}_{\geqslant 0}$ on a ring k is nothing else than a subadditive homomorphism of ordered semirings. If we exchange the usual addition of $\mathbb{R}_{\geqslant 0}$ by the maximum operation, which yields the ordered semiring $\mathbb{T}$ of tropical numbers, then a subadditive homomorphism $v : k \to \mathbb{T}$ is nothing else than a nonarchimedean seminorm on k.

Given a field k with a nonarchimedean absolute value $v : k \to \mathbb{T}$ and an affine k-scheme $X = \operatorname{Spec} R$, the Berkovich analytification X^{an} equals the set of all seminorms $w : R \to \mathbb{T}$ that extend v, i.e. the set of all subadditive homomorphisms w that make the diagram

$$\begin{tikzcd} k \arrow[r, "v"] \arrow[d] & \mathbb{T} \arrow[d, "\mathrm{id}"] \\ R \arrow[r, dashed, "w"] & \mathbb{T} \end{tikzcd}$$

commute. This suggests the interpretation of the Berkovich space X^{an} as the set $(X \otimes_k \mathbb{T})(\mathbb{T})$ of $\mathbb{T}$-rational points of the base change $X \otimes_k \mathbb{T} = \operatorname{Spec}(R \otimes_k \mathbb{T})$ of X along v.

The problem is that it is not clear if tensor products exist in general and how to construct them. We circumvent this problem by considering the larger category of ordered blueprints, which contains tensor products naturally; cf. Remarks 2.7 and 7.7 for details.

Ordered blueprints. In the following exposition, we present a different, but equivalent, definition of ordered blueprints from the main text of this paper. For the precise connection between these two viewpoints, cf. Remark 2.6.

An *ordered blueprint* is an ordered semiring B^+ together with a multiplicatively closed subset $B^\bullet \subset B^+$ of generators of B^+ that contains 0 and 1. A *morphism of blueprints* is an order preserving homomorphism of semirings that sends generators to generators. This defines the category OBlpr of ordered blueprints, which turns out to be closed under small limits and colimits and, in particular, has a tensor product. We write B for the ordered blueprint $B^\bullet \subset B^+$.

Some examples are the following. A semiring R can be considered as the ordered blueprint $B = (R \subset R)$, together with the trivial order on R. We call ordered blueprints B whose semiring is trivially ordered *algebraic*, and we can associate with every ordered blueprint B its *algebraic core* B^{core} which results from replacing the order of B^+ by the trivial order.

We denote the algebraic semiring $\mathbb{R}_{\geqslant 0}$ together with its natural total order by $\mathbb{R}_{\geqslant 0}^{\text{pos}}$. Similarly, we denote the algebraic semiring $\mathbb{T}$ of tropical numbers together with its natural total order by $\mathbb{T}^{\text{pos}}$. More generally, we can define for every ordered blueprint B its associated *totally positive blueprint* B^{pos}, which is B together with the order generated by the relations $a \leqslant b$ whenever there is an $c \in B^+$ such that $a + c \leqslant b$ in B^+. Note that it comes with a morphism $B \to B^{\text{pos}}$.

The name stems from the fact that $0 \leqslant a$ for every $a \in B^{\text{pos}}$. Note that in general, the order of B^{pos} might identify different elements of B^+. For instance, R^{pos} is trivial if R is a ring. For an idempotent semiring R, however, the totally positive blueprint R^{pos} carries the natural partial order of R and we recover R as $(R^{\text{pos}})^{\text{core}}$.

Valuations. In agreement with the naive approach explained above, totally positive blueprints will play the role of the recipients of valuations. Our interpretation of the domains of valuations will pass through the following construction.

Let $B = (B^\bullet \subset B^+)$ be an ordered blueprint. We define its associated *monomial blueprint* as the ordered blueprint $B^\bullet \subset B^{\text{mon},+}$ where $B^{\text{mon},+}$ is the monoid semiring $\mathbb{N}[B^\bullet]$ of $B^\bullet$ modulo the identification of the respective zeros of $B^\bullet$ and $\mathbb{N}[B^\bullet]$. The partial order of $B^{\text{mon},+}$ is generated by the *left monomial relations* $a \leqslant \sum b_j$ with $a, b_j \in B^\bullet$ whenever this holds in B^+. Note that the identity on $B^\bullet$ induces a morphism $B^{\text{mon}} \to B$.

With these definitions at hand, we see that a map $v : B \to \mathbb{R}_{\geqslant 0}$ from a ring B to the nonnegative reals is a seminorm if and only if the composition

$$B^{\text{mon}} \longrightarrow B \xrightarrow{v} \mathbb{R}_{\geqslant 0} \longrightarrow \mathbb{R}_{\geqslant 0}^{\text{pos}}$$

is a morphism of ordered blueprints. A map $v : B \to \mathbb{T}$ is a nonarchimedean seminorm if and only if $B^{\text{mon}} \to \mathbb{T}^{\text{pos}}$ is a morphism.

This motivates our general definition of a *valuation* as a multiplicative map $v : B \to T$ between ordered blueprints B and T such that $B^{\text{mon}} \to T^{\text{pos}}$ is a morphism.

Scheme theory. It is possible to extend ordered blueprints to a geometric category $\text{Sch}_{\mathbb{F}_1}$ of *ordered blue schemes* in terms of topological spaces with a sheaf in OBlpr. This comes with a contravariant functor $\text{Spec} : \text{OBlpr} \to \text{Sch}_{\mathbb{F}_1}$ that associates with an ordered blueprint the space of its prime k-ideals.

In particular, we can consider the sets $X(T)$ of T-rational points of an ordered blue T-scheme X. If T carries a topology, then $X(T)$ becomes a topological space with respect to the fine topology, which was introduced by the author and Salgado in [35].

Tropicalization and the bend relation. To avoid technicalities concerning scheme theory, we restrict ourselves to affine schemes in the following presentation of our results. This suffices to explain the essential content of our theory since tropicalization is a process that commutes with

restrictions to affine patches and a generalization to geometry is achieved by standard arguments in most situations.

Let k be an ordered blueprint and B an *ordered blue k-algebra*, i.e. a morphism $k \to B$. Let $v : k \to T$ be a valuation. Consider the functor $\mathrm{Val}_v(B,-)$ that associates with an ordered blue T-algebra S the set of valuations $w : B \to S$ that extend v, which means that the diagram

$$\begin{tikzcd} B \arrow[r, "w"] & S \\ k \arrow[u] \arrow[r, "v"] & T \arrow[u] \end{tikzcd}$$

commutes. Let $X = \mathrm{Spec}\, B$. A *tropicalization of X along v* is an ordered blue T-scheme that represents $\mathrm{Val}_v(B,-)$.

In complete generality, a tropicalization of X along v does not exist. However, in the following two situations, we can prove its existence in terms of an explicit description. Let $X^{\mathrm{mon}} = \mathrm{Spec}\, B^{\mathrm{mon}}$.

Theorem A. *If T is totally positive, then $X^{\mathrm{mon}} \otimes_{k^{\mathrm{mon}}} T$ is a tropicalization of X along v.*

This is Theorem 7.4. This theorem realizes the idea that the tropicalization is the base change along a valuation. In particular, if k and B are monomial, then $X \otimes_k T$ is a tropicalization of X along v.

In order to formulate the second existence theorem for tropicalizations, we have to introduce the bend, which is a generalization of the Giansiracusa tropicalization ([21]) to the context of ordered blueprints. The *bend of an ordered blueprint B along v* is the ordered blue T-algebra $\mathrm{Bend}_v(B)$ whose ordered semiring $\mathrm{Bend}_v(B)^+$ and whose underlying monoid $\mathrm{Bend}_v(B)^\bullet$ are defined as follows. The semiring $\mathrm{Bend}_v(B)^+$ is the quotient of the semigroup semiring $T^+[B^\bullet]$ by the relations of the form

$$(v(c)t)\cdot a \;=\; t\cdot(c.a) \qquad \text{and} \qquad t\cdot a + \sum t\cdot b_j \;=\; \sum t\cdot b_j$$

with $c \in k^\bullet$, $t \in T^\bullet$, $a, b_j \in B^\bullet$ and $a \leqslant \sum b_j$ in B^+. The monoid $\mathrm{Bend}_v(B)^\bullet$ consists of the classes of elements of the form $t\cdot a$ in $\mathrm{Bend}_v(B)^+$ and the order of $\mathrm{Bend}_v(B)^+$ is generated by the order of T.

We say that T is *idempotent* if T^+ is an idempotent semiring. For $X = \mathrm{Spec}\, B$, we define $\mathrm{Bend}_v(X) = \mathrm{Spec}\,\mathrm{Bend}_v(B)$. The following is Theorem 7.16.

Theorem B. *If T is idempotent, then there is a canonical isomorphism*

$$\mathrm{Bend}_v(B) \quad \xrightarrow{\sim} \quad (B^{\mathrm{mon}} \otimes_{k^{\mathrm{mon}}} T^{\mathrm{pos}})^{\mathrm{core}} \otimes_{T^{\mathrm{core}}} T,$$

and $\mathrm{Bend}_v(X)$ is a tropicalization of X along v.

As a consequence, a tropicalization of X along v is algebraic and equal to the spectrum of $(B^{\mathrm{mon}} \otimes_{k^{\mathrm{mon}}} T^{\mathrm{pos}})^{\mathrm{core}}$ if T is idempotent and algebraic. If T is idempotent and totally positive, then $\mathrm{Bend}_v(B) = B^{\mathrm{mon}} \otimes_{k^{\mathrm{mon}}} T$.

In the following, we will explain how the different concepts of analytification and tropicalization of classical schemes fit into the framework of tropicalizations of ordered blueprints and ordered blue schemes.

Berkovich analytification and Kajiwara-Payne tropicalization. Let k be a field and $v : k \to \mathbb{T}$ a valuation. Let $Y = \mathrm{Spec}\, R$ a k-scheme and $\iota : Y \to \mathrm{Spec}\, k[A_0]$ a closed embedding into a toric k-variety.

The restriction of a seminorm $w : k[A_0] \to \mathbb{T}$ in Y^{an} to A_0 is a multiplicative map $A_0 \to \mathbb{T}$. If we define $\mathrm{Hom}(A_0, \mathbb{T})$ with the real topology coming from $\mathbb{T}$, then this restriction defines a continuous map $\mathrm{trop}^{KP}_{v,\iota} : Y^{\mathrm{an}} \to \mathrm{Hom}(A_0, \mathbb{T})$. The *Kajiwara-Payne tropicalization of Y* is the image $\mathrm{Trop}^{KP}_{v,\iota}(Y) = \mathrm{trop}^{KP}_{v,\iota}(Y^{\mathrm{an}})$ under this map.

The associated blueprint B is defined as $B^+ = R$ and $B^\bullet = \{\Gamma\iota(ca) | c \in k^\bullet, a \in A_0\}$ where $\Gamma\iota : k[A_0] \to R$ is the map of global sections induced by ι.

The inclusion $B \to R$ defines a morphism $\beta : Y \to Z$ of ordered blue schemes where $Z = \operatorname{Spec} B$. The following summarizes Theorems 8.1 and 9.1.

Theorem C. *The Berkovich space Y^{an} is naturally homeomorphic to $\mathrm{Bend}_v(Y)(\mathbb{T})$, the Kajiwara-Payne tropicalization $\mathrm{Trop}^{KP}_{v,\iota}(Y)$ is naturally homeomorphic to $\mathrm{Bend}_v(Z)(\mathbb{T})$ and the diagram*

$$\begin{tikzcd} Y^{\mathrm{an}} \arrow[r, "\mathrm{trop}^{KP}_{v,\iota}"] \arrow[d, "\simeq"] & \mathrm{Trop}^{KP}_{v,\iota}(Y) \arrow[d, "\simeq"] \\ \mathrm{Bend}_v(Y)(\mathbb{T}) \arrow[r, "\mathrm{Bend}_v(\beta)(\mathbb{T})"] & \mathrm{Bend}_v(Z)(\mathbb{T}) \end{tikzcd}$$

of continuous maps commutes.

Foster-Ranganathan tropicalization. Let $\mathbb{T}^{(n)} = \mathbb{R}^n_{>0} \cup \{0\}$ be the idempotent semiring with componentwise multiplications and whose addition is defined as taking the maximum with respect to the lexicographical order. With respect to the order topology, it is a topological Hausdorff semifield.

Let k be a field, endowed with a higher rank valuation $v : k \to \mathbb{T}^{(n)}$. Let $Y = \operatorname{Spec} R$ be an affine k-scheme and $\iota : Y \to \operatorname{Spec}[A_0]$ a closed immersion into a toric k-variety.

Replacing $\mathbb{T}$ by $\mathbb{T}^{(n)}$ in the definitions of the Berkovich analytification and the Kajiwara-Payne tropicalization yields the *Foster-Ranganathan analytification* $\mathrm{An}^{FR}_v(Y)$ *of Y along v* and the *Foster-Ranganathan tropicalization* $\mathrm{Trop}^{FR}_{v,\iota}(Y)$ *of Y along v with respect to ι*, respectively.

Let B be the blueprint associated with ι as defined above and $Z = \operatorname{Spec} B$. Let $Y \to Z$ be the induced morphism of blue k-schemes. The following is Theorem 10.1.

Theorem D. *The Foster-Ranganathan analytification $\mathrm{An}^{FR}_v(Y)$ is naturally homeomorphic to $\mathrm{Bend}_v(Y)(\mathbb{T}^{(n)})$, the Foster-Ranganathan tropicalization $\mathrm{Trop}^{FR}_{v,\iota}(Y)$ is naturally homeomorphic to $\mathrm{Bend}_v(Z)(\mathbb{T}^{(n)})$ and the diagram*

$$\begin{tikzcd} \mathrm{An}^{FR}_v(Y) \arrow[r, "\mathrm{trop}^{FR}_{v,\iota}"] \arrow[d, "\simeq"] & \mathrm{Trop}^{FR}_{v,\iota}(Y) \arrow[d, "\simeq"] \\ \mathrm{Bend}_v(Y)(\mathbb{T}^{(n)}) \arrow[r, "\mathrm{Bend}_v(\beta)(\mathbb{T}^{(n)})"] & \mathrm{Bend}_v(Z)(\mathbb{T}^{(n)}) \end{tikzcd}$$

of continuous maps commutes.

Giansiracusa tropicalization and Maclagan-Rincón weights. We remarked already that the bend of an ordered blueprint is a generalization of the Giansiracusa tropicalization from [21]. The precise relation is as follows.

Let k be a ring and $v : k \to T$ be a valuation into a totally ordered idempotent semiring T. Let $Y = \operatorname{Spec} R$ be a k-scheme. Let A_0 be a monoid and $\eta : A_0 \to R$ a multiplicative map such that $k[A_0] \to R$ is surjective. The *Giansiracusa tropicalization* $\mathrm{Trop}^{GG}_{v,\eta}(Y)$ *of Y with respect to v and η* is the spectrum of the semiring

$$\mathrm{Trop}^{GG}_{v,\eta}(R) \quad = \quad T[A_0] \, / \, \Big\{ \, a + \sum b_j \equiv \sum b_j \, \Big| \, \eta(a) + \sum \eta(b_j) = 0 \text{ in } R \, \Big\}.$$

The blueprint associated with η is $B = (A \subset R)$ where $A = \{c.\eta(a) \in R | c \in k, a \in A_0\}$. The following is Theorem 11.2.

Theorem E. *There is a canonical isomorphism $\mathrm{Trop}^{GG}_{v,\eta}(R) \simeq \mathrm{Bend}_v(B)^+$ of semirings.*

The Giansiracusa tropicalization $\mathrm{Trop}^{GG}_{v,\eta}(Y)$ comes with a closed embedding into the toric $\mathbb{T}$-scheme $\operatorname{Spec} \mathbb{T}[A_0]$. Maclagan and Rincón ([36, Thm. 1.2]) show that the structure of the

tropical variety $\mathrm{Trop}(Y) = \mathrm{Trop}^{GG}_{v,\eta}(Y)(\mathbb{T})$ as a weighted polyhedral complex can be recovered from this embedding, assuming the following context: $v : k \to \mathbb{T}$ is a valuation with dense image such that the value group $v(k^\times)$ lifts to $k^\times$ and assume that $Y = \mathrm{Spec}\, R$ is an equidimensional closed k-subvariety of $\mathbb{G}^n_{m,k}$, which corresponds to a multiplicative map $\eta : A_0 \to R$ where $A_0 = \{X_1^{e_1} \cdots X_n^{e_n} | (e_1, \dots, e_n) \in \mathbb{Z}^n\}$.

We show that in this situation, the structure of a weighted polyhedral complex can still be recovered from the weaker structure of the associated blue $\mathbb{T}$-scheme $\mathrm{Bend}_v(Z)$. More precisely, we exhibit an explicit formula for the *Maclagan-Rincón weight* $\mu(w)$ of a $\mathbb{T}$-rational point w of $\mathrm{Bend}_v(Z)$ and show the following in Theorem 12.3.

Theorem F. *Let σ be a top dimensional polyhedron of* $\mathrm{Trop}(Y) = \mathrm{Bend}_v(Z)(\mathbb{T})$*. Then* $\mathrm{mult}(\sigma) = \mu(w)$ *for every w in the relative interior of σ.*

Macpherson analytification. Let k be a ring and B a k-algebra. The *Macpherson analytification of B over k* is the idempotent semiring $\mathrm{An}(B,k)$ of finitely generated k-submodules M_1 and M_2 of B with respect to the addition $M_1 + M_2$ and the multiplication $M_1 \cdot M_2$ given by elementwise operations. The semiring $\mathrm{An}(B,k)$ represents the functor $\mathrm{Val}(B,k;-)$ that associates with an idempotent semiring T the set of all valuations $v : B \to T$ that are integral on k, i.e. $v(c) + 1 = 1$ for all $c \in k$.

This concept can be generalized to any ordered blue k-algebra B over an ordered blueprint k. A k-span of B is a subset M of B that is stable under multiplication by $k^\bullet$ and contains all $b \in B$ for which there are elements $a_i \in M$ and a relation $b \leqslant \sum a_i$ in B^+. A k-span is finitely generated if it contains a finite subset such that there is no smaller k-span containing it. We define $\mathrm{An}(B,k)$ as the idempotent semiring of all finitely generated k-spans of B.

The definition of $\mathrm{Val}(B,k;-)$ extends to this setting as a functor on idempotent semirings, which are the same as $\mathbb{B}$-algebras where $\mathbb{B}$ is the Boolean semifield.

In order to describe $\mathrm{An}(B,k)$ as a bend, we define $B^{\mathrm{mon}}_{k\leqslant 1}$ as the ordered blueprint B^{mon} together with the order that contains all relations of B^{mon} together with the relations $a.1 \leqslant 1$ for $a \in k$. Define $\mathbb{F}_1 = (\{0,1\} \subset \mathbb{N})$, together with the trivial order. There is a unique valuation $v_0 : \mathbb{F}_1 \to \mathbb{B}$, given by $v_0(0) = 0$ and $v_0(a) = 1$ for $a > 0$. The following is Theorem 13.2.

Theorem G. *There is a canonical isomorphism* $\mathrm{An}(B,k) \simeq \mathrm{Bend}_{v_0}(B^{\mathrm{mon}}_{k\leqslant 1})^+$ *of semirings and* $\mathrm{An}(B,k)$ *represents* $\mathrm{Val}(B,k;-)$.

Let $v : k \to T$ be a valuation of a ring k into an idempotent semiring and $\mathcal{O}_k = \{a \in k | v(a) + 1 = 1\}$, which is a subring of k. As a consequence of Theorem G, we obtain $\mathrm{Bend}_v(B)^+ \simeq \mathrm{An}(B, \mathcal{O}_k) \otimes_{\mathrm{An}(k,\mathcal{O}_k)} T$ in Corollary 13.4. This provides an alternative to Macpherson's original construction of tropicalizations via nonarchimedean analytic geometry, cf. [38, section 7.3].

Thuillier analytification and Ulirsch tropicalization. Let k be a field and $v : k \to \mathcal{O}_\mathbb{T}$ the trivial valuation where $\mathcal{O}_\mathbb{T}$ is the subsemiring $\{a \in \mathbb{T} | a + 1 = 1\}$ of $\mathbb{T}$. We consider $\mathcal{O}_\mathbb{T}$ together with its topology as a subset of $\mathbb{R}$. Let $X = \mathrm{Spec}\, R$ be a k-scheme. The Thuillier analytification $X^\beth$ is the set of all extensions $w : R \to \mathcal{O}_\mathbb{T}$ of v to R, together with the topology induced by $\mathcal{O}_\mathbb{T}$.

Let $\alpha : \mathcal{M}_X \to \mathcal{O}_X$ be a fine and saturated log structure for X without monodromy. Then there is a universal morphism $(X, \mathcal{M}_X / \mathcal{M}_X^\times) \to F_X$ of monoidal spaces into a Kato fan F_X, and this morphism induces a continuous map $\mathrm{trop}^U_\alpha : X^\beth \to \overline{\Sigma}_X$ into the extended cone complex $\overline{\Sigma}_X$ of $\mathcal{O}_\mathbb{T}$-rational points of the Kato fan F_X. The Ulirsch tropicalization of a closed k-subscheme $Y = \mathrm{Spec}\, S$ of X is the image $\mathrm{Trop}^U_{\alpha,\iota}(Y) = \mathrm{trop}^U_\alpha(Y^\beth)$ in $\overline{\Sigma}_X$ where ι refers to the closed immersion $\iota : Y \to X$.

For the sake of simplifying this exposition, we assume that the Kato fan is affine. We define the associated blue k-scheme Z as the spectrum of the blueprint B where $B^\bullet$ is the image of $\mathcal{M}_X(X)$ under $\Gamma\iota : R \to S$ and B^+ is the subsemiring of S generated by $B^\bullet$. The blue scheme Z

comes together with a morphism $\beta : Y \to Z$. Then the following summarizes Theorems 14.1, 15.6 and 15.10.

Theorem H. *The Thuillier space* $X^{\beth}$ *is naturally homeomorphic to* $\mathrm{Bend}_v(X)(\mathcal{O}_{\mathbb{T}})$*, the Ulirsch tropicalization* $\mathrm{Trop}^U_{\alpha,\iota}(Y)$ *is naturally homeomorphic to the topological space* $\mathrm{Bend}_v(Z)(\mathcal{O}_{\mathbb{T}})$ *and the diagram*

$$\begin{tikzcd} Y^{\beth} \arrow[rr, "\mathrm{trop}^U_{\alpha,\iota}(Y)"] \arrow[d, "\simeq"] & & \mathrm{Trop}^U_{\alpha,\iota}(Y) \arrow[d, "\simeq"] \\ \mathrm{Bend}_v(Y)(\mathcal{O}_{\mathbb{T}}) \arrow[rr, "\mathrm{Bend}_v(\beta)(\mathcal{O}_{\mathbb{T}})"] & & \mathrm{Bend}_v(Z)(\mathcal{O}_{\mathbb{T}}) \end{tikzcd}$$

of continuous maps commutes. If $Y = X$ *and* $\alpha : \mathcal{M}_X \to \mathcal{O}_X$ *is a monomorphism of sheaves, then we can recover the Kato fan* F_X *and the embedding* $\mathrm{Trop}^U_{\alpha,\iota}(X) \to \overline{\Sigma}_X$ *from* $\mathrm{Bend}_v(Z)$.

Note that the restriction to the trivial valuation $v : k \to \mathcal{O}_T$ is caused by the following technical obstruction: the Ulirsch tropicalization relies on the choices of local sections to $\mathcal{M}_X \to \mathcal{M}_X/\mathcal{M}_X^{\times}$, and this contribution can be avoided since $\mathcal{O}_{\mathbb{T}}^{\times} = \{1\}$. Passing from the fine and saturated log structure to the associated blue scheme avoids these choices and overcomes any restrictions on the valuation.

Conclusion. The scheme theoretic tropicalization in terms of blue schemes provides a framework that embraces all other concepts of tropicalization considered in this paper, up to some technical restrictions that we address below. This theory extends the different generalizations of the Kajiwara-Payne tropicalization commonly into all directions, with exception of the restrictions mentioned below. In particular, this means that the scheme theoretic structure of the Giansiracusa tropicalization extends to the context of Thuillier analytification and Ulirsch tropicalization of fine and saturated log schemes with respect to any valuation $v : k \to T$ into idempotent T.

Another more subtle improvement is the following. The tropicalization of a blue scheme comes with the structure of a blue scheme. This additional structure determines the tropical variety as a topological space, and in the case of a closed subscheme of a torus, it encodes the weights that appear if the tropical variety is gets identified with a polyhedral weighted complex. This has the consequence that we can detach the tropicalization from its ambient space like a toric variety or an extended cone complex.

Technical restrictions. In this text, we do not pursue a theory of étale morphisms for blue schemes. Therefore we restrict ourselves to Zariski log schemes in our treatment of Ulirsch tropicalization. We further assume that the continuous map $\chi : X \to F_X$ from the log-scheme X to its Kato fan satisfies that the inverse image of an affine open subset is affine.

We expect that these restrictions are not essential, but we leave the treatment of a more general theory and, in particular, étale morphisms for ordered blue schemes to future investigations.

Content overview. This text is divided into two parts. The first part introduces ordered blueprints and ordered blue schemes. The second part applies this theory to tropicalization.

The first part contains the following sections. After settling some conventions for this paper in section 1, we introduce ordered blueprints and various subcategories and functorial constructions in section 2. In section 3, we explain the relation of our notion of valuations to seminorms and Krull valuations. In section 4, we review and extend the theory of blue schemes to the realm of ordered blueprints, which provides the scheme theoretic background for the constructions in the second part of the paper. In section 5, we explain how a topology on an ordered blueprint T yields a topology on the set $X(T)$ of T-rational points. In section 6, we explain the connections to relative schemes after Toën and Vaquié and comment on the differences to the earlier version [34] of this paper.

In section 7, we introduce the general concept of tropicalizing an ordered blue scheme along a valuation and prove the central results Theorems A and B. In the subsequent sections, we explain the relation to other concepts of analytifications and tropicalizations and prove Theorems C–H. Since the section headers are self-explanatory, we refer the reader to the table of contents for finding the corresponding results.

Acknowledgements. I would like to thank Sam Payne for organizing the meeting Algebraic Foundations for Tropical Geometry in May 2014. I would like to thank all participants for our discussions during the workshop. My particular thanks go to Matt Baker and Andrew Macpherson for their patient explanations on analytic geometry and skeleta; to Jeff and Noah Giansiracusa for our conversations on scheme theoretic foundations; to Diane Maclagan for sharing her ideas on tropical schemes and her comments on a previous version of this text; and to Martin Ulirsch for our discussions on the connection between blue schemes and log schemes, and for his careful corrections of a previous version. I would like to thank Ethan Cotterill for bringing several publications to my attention and for his help with various questions on tropical geometry. I would like to thank Walter Gubler, Joseph Rabinoff and Annette Werner for their explanations on skeleta, which led me to the conclusion that the theory of this paper is not yet sufficiently developed to explain skeleta as rational point sets of underlying schemes. I would like to thank Foster Tyler and Dhruv Ranganathan for their explanations on higher rank analytifications and tropicalizations.

Part 1. Ordered blueprints

In the first part, we set up the theory of ordered blueprints and ordered blue schemes. The category of ordered blueprints recovers well-known objects as ordered semirings and monoids as well as blueprints, halos, hyperrings and sesquiads. Several constructions of endofunctors allow us to talk about seminorms and valuations in terms of morphisms. Section 2.10 contains an illustration of the relevant subcategories of the category of ordered blueprints.

The spectrum of an ordered blueprint is based on the notion of a prime k-ideal, which yields a topological space together with a structure sheaf. This lets us define an ordered blue scheme as a so-called ordered blueprinted space that is covered by spectra of ordered blueprints. After explaining how a topology for an ordered blueprint T induces a topology for the set $X(T)$ of T-rational points of an ordered blue scheme X, we briefly sketch how the approach to ordered blue schemes is connected to Toën and Vaquié's relative schemes, and comment on the gap originating from different Grothendieck topologies on the category of ordered blueprints.

1. Conventions

In this text we use the following conventions. A *monoid* is a multiplicatively written commutative semigroup A with unit element 1 and a morphism of monoids is a multiplicative map that maps 1 to 1. A *monoid with zero* is a monoid A with an additional element 0 that satisfies $0\cdot a = 0$ for all $a \in A$. A morphism of monoids with zero is a morphism of monoids that maps 0 to 0. We denote the category of monoids with zero by Mon.

A *semiring* is always commutative with 0 and 1. A ring is a semiring with an additive inverse -1 of 1. An idempotent semiring is a semiring with $1+1=1$.

The reader that is familiar with blueprints will find that the definition in section 2.3 is equivalent to the definitions in other texts on blueprints, like [31] and [33], with the exception of [31] where a blueprint in the sense of this text is called a *proper blueprint with zero*.

As a last point, we like to draw the reader's attention to the following inconsistency with standard notation, as used in the introduction. Tensor products and free objects of (ordered) semirings, considered in the category of (ordered) blueprints, are not (ordered) semirings. Our

convention is to use the according standard symbols for the constructions *in the category of ordered blueprints* and to refer to the corresponding construction inside the category of ordered semirings with a superscript "$+$". This applies to the following notations.

Given a blueprint B and a monoid with zero A, we write $B[A]$ for the free blueprint over B, whose underlying set is $\{0\} \cup \{b \cdot a | b \in B, a \in A\}$. We write $B[A]^+$ for the generated semiring. For instance, while we denote by $\mathbb{N}[A]^+ = \{\sum a_i T^i | a_i \in \mathbb{N}\}$ the semiring of polynomials, we denote by $\mathbb{N}[A]$ the blueprint of monomials aT^i with $a \in \mathbb{N}$. Given semiring homomorphisms $D \to B$ and $D \to C$, we denote by $B \otimes_D^+ C$ the tensor product in the category of semirings, which differs in general from the tensor product $B \otimes_D C$ in the category of blueprints. Note that the precise relationship is given by $B \otimes_D^+ C = (B \otimes_D C)^+$.

Another instance of this notation are the affine line and the multiplicative group scheme. In the category of (ordered) blueprints, the functor $B \mapsto B$ is represented by $\mathbb{A}_B^1 = \operatorname{Spec} B[X]$ and the functor $B \to B^\times$ is represented by $\mathbb{G}_{m,B} = \operatorname{Spec} B[X^{\pm 1}]$. To distinct these objects from the classical affine line and the classical multiplicative group scheme for a semiring B, we use $\mathbb{A}_B^{1,+} = \operatorname{Spec} B[X]^+$ and $\mathbb{G}_{m,B}^+ = \operatorname{Spec} B[X^{\pm 1}]^+$ for the latter objects.

2. Basic definitions

In this section, we introduce the category of ordered blueprints and various subcategories and endofunctors that are of relevance for this paper.

Definition 2.1. An *ordered blueprint* is a monoid A with zero together with a *subaddition on A*, which is a relation $\mathcal{R}$ on the set $\mathbb{N}[A]^+ = \{\sum a_i | a_i \in A\}$ of finite formal sums of elements of A that satisfies the following list of axioms (where we write $\sum a_i \leqslant \sum b_j$ for $(\sum a_i, \sum b_j) \in \mathcal{R}$ with $a_i, b_j \in A$, and where 0 is the zero in A and (empty sum) is the empty sum in $\mathbb{N}[A]^+$).

- (B1) $a \leqslant a$ for all $a \in A$; *(reflective)*
- (B2) $\sum a_i \leqslant \sum b_j$ and $\sum b_j \leqslant \sum c_k$ implies $\sum a_i \leqslant \sum c_k$; *(transitive)*
- (B3) $\sum a_i \leqslant \sum b_j$ and $\sum c_k \leqslant \sum d_l$ implies $\sum a_i + \sum c_k \leqslant \sum b_j + \sum d_l$; *(additive)*
- (B4) $\sum a_i \leqslant \sum b_j$ and $\sum c_k \leqslant \sum d_l$ implies $\sum a_i c_k \leqslant \sum b_j d_l$; *(multiplicative)*
- (B5) $0 \leqslant$ (empty sum) and (empty sum) $\leqslant 0$; *(zero)*
- (B6) $a \leqslant b$ and $b \leqslant a$ implies $a = b$ as elements in A. *(proper)*

We write $B = A /\!\!/ \mathcal{R}$ for an ordered blueprint with A and $\mathcal{R}$ as above. A *morphism of ordered blueprints $B_1 = A_1 /\!\!/ \mathcal{R}_1$ and $B_2 = A_2 /\!\!/ \mathcal{R}_2$* is a monoid morphism $f : A_1 \to A_2$ such that $\sum a_i \leqslant_1 \sum b_j$ implies $\sum f(a_i) \leqslant_2 \sum f(b_j)$. We denote the category of ordered blueprints by OBlpr.

Note that axioms (B1) and (B2) state that $\mathcal{R}$ is a *pre-order* on $\mathbb{N}[A]^+$, and axiom (B6) states that $\mathcal{R}$ restricts to a partial order on A, considered as a subset of $\mathbb{N}[A]^+$.

Often, we refer to the underlying set of A by the symbol B, i.e. we write $a \in B$ for $a \in A$ and $f : B_1 \to B_2$ for the underlying map $A_1 \to A_2$ of monoids. We say that $\sum a_i \leqslant \sum b_j$ holds in $B = A /\!\!/ \mathcal{R}$ if $(\sum a_i, \sum b_j)$ is an element of $\mathcal{R}$. We write $\sum a_i \geqslant \sum b_j$ if $\sum b_j \leqslant \sum a_i$, and $\sum a_i \equiv \sum b_j$ if both $\sum a_i \leqslant \sum b_j$ and $\sum a_i \geqslant \sum b_j$. For instance, axiom (B5) can be rewritten as $0 \equiv$ (empty sum).

Any set S of relations (of the form $\sum a_i \leqslant \sum b_j$) has a closure $\langle S \rangle$ under axioms (B1)–(B5), which is the smallest relation $\mathcal{R}$ on $\mathbb{N}[A]^+$ that contains S and satisfies axioms (B1)–(B5). However, axiom (B6) plays a restrictive role: not every set S of relations is contained in a relation $\mathcal{R}$ on $\mathbb{N}[A]^+$ that satisfies (B6). Therefore $\langle S \rangle$ will always refer to the *closure under (B1)–(B5)*, while $B = A /\!\!/ \langle S \rangle$ refers to the proper quotient, as introduced in the following section.

Every monoid A with zero has a smallest subaddition $\langle \emptyset \rangle$, and we consider A as the ordered blueprint $A /\!\!/ \langle \emptyset \rangle$. Since a map $A_1 \to A_2$ is a morphism in Mon if and only it is a morphism in OBlpr, this defines a full embedding Mon $\to$ OBlpr. We say that an ordered blueprint B is a *monoid* if it is in the essential image of this embedding.

2.1. Proper quotients. Given a pair $B = (A, \mathcal{R})$ that satisfies axioms (B1)–(B5), we can associate with it the following ordered blueprint $B_{\text{prop}} = (A/\sim)/\!\!/\widetilde{\mathcal{R}}$. We define the equivalence relation $\sim$ on A by $a \sim b$ if $a \equiv b$. We define $\sum \bar{a}_i \widetilde{\leqslant} \sum \bar{b}_j$ if $\sum a_i \leqslant \sum b_j$ where $\bar{a}_i, \bar{b}_j \in (A/\sim)$ are the respective classes of $a_i, b_j \in A$. By axiom (B5), $A/\sim$ is a monoid with zero, and axiom (B2) ensures that the definition of $\widetilde{\mathcal{R}}$ is independent of the choice of representatives. It is easily verified that B_{prop} is indeed an ordered blueprint, which we call the *proper quotient of* B; we say that $B = (A, \mathcal{R})$ is an *(improper) representation of* B_{prop}. We denote by $A/\!\!/\mathcal{R}$ the proper quotient of $(A, \mathcal{R})$.

Let $B = (A, \mathcal{R})$ be as above and $f : B \to C$ a multiplicative map into an ordered blueprint C such that $\sum a_i \leqslant \sum b_j$ in B implies $\sum f(a_i) \leqslant \sum f(b_j)$ in C. Then f factors uniquely into the quotient map $B \to B_{\text{prop}}$ followed by a morphism $f_{\text{prop}} : B_{\text{prop}} \to C$. This shows that $(-)_{\text{prop}}$ is functorial in pairs $(A, \mathcal{R})$ as above. If we want to stress that $(A, \mathcal{R})$ is a proper representation of B, then we write that $B = (A, \mathcal{R})$ is an ordered blueprint.

Let $B = A/\!\!/\mathcal{R}$ be an ordered blueprint and $\sim$ the restriction of $\mathcal{R}$ to A. Then we call $B^\bullet = A/\sim$ the *underlying monoid* $A/\sim$ of $A/\!\!/\mathcal{R}$. Note that a morphism $B \to C$ determines a monoid morphism $f^\bullet : B^\bullet \to C^\bullet$ between the underlying monoids of B and C. This yields a functor $(-)^\bullet : \text{OBlpr} \to \text{Mon}$, which is right adjoint and left inverse to the embedding $\text{Mon} \to \text{OBlpr}$.

We say that a morphism $f : B \to C$ is *injective* or *surjective* if the map $f^\bullet : B^\bullet \to C^\bullet$ between the underlying monoids is injective or surjective, respectively. A morphism $f : B \to C$ of blueprints is *full* if every relation $\sum f(a_i) \leqslant \sum f(b_j)$ in C with $a_i, b_j \in B$ implies $\sum a_i \leqslant \sum b_j$ in B.

A *subblueprint of* B is a blueprint B' together with an injective morphism $B' \to B$. Note that a morphism is a monomorphism if and only if the map between the underlying monoids is injective. A subblueprint is *full* if the inclusion $B' \to B$ is full. Note that a full subblueprint $B' \subset B$ is determined by the submonoid $(B')^\bullet$ of $B^\bullet$.

Let $B = A/\!\!/\mathcal{R}$ be a blueprint and S be a set of relations on $\mathbb{N}[A]^+$. We denote by $B/\!\!/\langle S \rangle$ the blueprint $A/\!\!/\langle \mathcal{R} \cup S \rangle$.

2.2. Limits and colimits. The product $B = A/\!\!/\mathcal{R}$ of a family of blueprints $B_l = A_l/\!\!/\mathcal{R}_l$ is represented by the Cartesian product $A = \prod A_l$ of the underlying monoids with coordinatewise multiplication, together with the subaddition

$$\mathcal{R} \quad = \quad \{ \sum a_i \leqslant \sum b_j \mid \sum a_{i,l} \leqslant \sum b_{j,l} \text{ for all } l \}.$$

The equalizer $\text{eq}(f, g)$ of two morphisms $f, g : B_1 \to B_2$ is represented by the full ordered subblueprint $B = A/\!\!/\mathcal{R}$ of B_1 with

$$A \quad = \quad \{ a \in B_1 \mid f(a) = g(a) \}.$$

The coequalizer of two morphisms $f_1, f_2 : B_1 \to B_2$ is $A/\!\!/\mathcal{R}$ where A is the underlying monoid of B_2 and $\mathcal{R}$ is generated by the subaddition of B_2 and all relations of the form $f_1(a) \equiv f_2(a)$ with $a \in B_1$.

The coproduct of two ordered blueprints B and C is the smash product $B \wedge C$, which is obtained from Cartesian product $B \times C$ by identifying $B \times \{0\} \cup \{0\} \times C$ with 0. In particular, there is a tensor product $B \otimes_D C$ for every diagram $B \leftarrow D \to C$, which is the quotient of $B \times C$ by all relations of the form $(db, c) = (b, dc)$ where $b \in B$, $c \in C$ and $d \in D$. The coproduct of an infinite family is the filtered colimit of the coproducts of its finite subfamilies. The filtered colimit can be constructed as usual; for instance, see [12].

An initial object of OBlpr is the monoid $\mathbb{F}_1 = \{0, 1\}$ and a terminal object is the *trivial blueprint* $0 = \{0\}/\!\!/\langle\emptyset\rangle$. We summarize:

Lemma 2.2. *The category of ordered blueprints is complete and cocomplete with initial and terminal objects.* □

2.3. Algebraic blueprints. Whenever we see the need to make a clear distinction between blueprints as considered in [31] and other types of blueprints as they appear in this text, we shall call them algebraic blueprints.

An *(algebraic) blueprint* is a pair of a monoid A together with a *preaddition*, which is a subaddition $\mathcal{R}$ that satisfies

(B7) $\sum a_i \leqslant \sum b_j$ if and only if $\sum b_j \leqslant \sum a_i$, *(symmetric)*

i.e. $\mathcal{R}$ is an equivalence relation on $\mathbb{N}[A]^+$ that satisfies the additional axioms (B3)–(B6). In other words, an ordered blueprint $B = A /\!\!/ \mathcal{R}$ is an algebraic blueprint if and only if $\mathcal{R}$ is symmetric.

A *morphism of algebraic blueprints* is the same as a morphism of ordered blueprints. We denote the full subcategory of algebraic blueprints in OBlpr by $\text{Blpr} = \text{OBlpr}^{\text{alg}}$ and the embedding as full subcategory by

$$\iota^{\text{alg}}: \quad \text{Blpr} \quad \longrightarrow \quad \text{OBlpr}.$$

The embedding ι^{alg} has a left adjoint

$$(-)^{\text{hull}}: \quad \text{OBlpr} \quad \longrightarrow \quad \text{Blpr},$$

which sends an ordered blueprint $B = A /\!\!/ \mathcal{R}$ to its *algebraic hull* $B^{\text{hull}} = A /\!\!/ \mathcal{R}^{\text{hull}}$ with

$$\mathcal{R}^{\text{hull}} \quad = \quad \{ \sum a_i \equiv \sum b_j \mid \sum a_i \leqslant \sum b_j \text{ in } B \}.$$

Note that $(A, \mathcal{R}^{\text{hull}})$ is in general not a proper representation of B^{hull}, even if $(A, \mathcal{R})$ is a proper representation of B. The algebraic hull comes with a canonical morphism $B \to B^{\text{hull}}$ that maps $a \in B$ to its class in the proper quotient of $(A, \mathcal{R}^{\text{hull}})$. This morphism is universal among all morphisms from B to algebraic blueprints, which explains the functoriality of $(-)^{\text{hull}}$.

The embedding ι^{alg} has also a right adjoint

$$(-)^{\text{core}}: \quad \text{OBlpr} \quad \longrightarrow \quad \text{Blpr},$$

which sends an ordered blueprint $B = A /\!\!/ \mathcal{R}$ to its *algebraic core* $B^{\text{core}} = A /\!\!/ \mathcal{R}^{\text{core}}$ with

$$\mathcal{R}^{\text{core}} \quad = \quad \{ \sum a_i \equiv \sum b_j \mid \sum a_i \equiv \sum b_j \text{ in } B \}.$$

Note that $(A, \mathcal{R}^{\text{core}})$ is always a proper representation of B^{core} if $(A, \mathcal{R})$ is a proper representation of B. The algebraic core comes with a canonical morphism $B^{\text{core}} \to B$ that is the identity on the underlying monoid A. This morphism is universal among all morphisms from an algebraic blueprint to B, which explains the functoriality of $(-)^{\text{core}}$.

Lemma 2.3. *An ordered blueprint B is an algebraic blueprint if and only if one of the following equivalent conditions are satisfied:*

(i) *The canonical morphism $B \to B^{\text{hull}}$ is an isomorphism.*
(ii) *The canonical morphism $B^{\text{core}} \to B$ is an isomorphism.*
(iii) *The canonical morphism $B^{\text{core}} \to B^{\text{hull}}$ is an isomorphism.*

Proof. This follows easily from the definitions. □

2.4. Blueprints with inverses. A *blueprint with* -1 (or *with inverses*) is an ordered blueprint a blueprint B that contains an element -1 that satisfies $1 + (-1) \equiv 0$. This implies that every element $a \in B$ has an additive inverse, which is an element $-a$ with $a + (-a) \equiv 0$. The additive inverse is necessarily unique.

The blueprint $\mathbb{F}_{1^2} = \{0, \pm 1\} /\!\!/ \langle 1 + (-1) \equiv 0 \rangle$ has a unique morphism into any other blueprint with -1. Therefore the full subcategory of blueprints with -1 corresponds to $\text{Blpr}_{\mathbb{F}_{1^2}}$, and the base change functor $- \otimes_{\mathbb{F}_1} \mathbb{F}_{1^2} : \text{OBlpr} \to \text{Blpr}_{\mathbb{F}_{1^2}}$ is left adjoint and left inverse to the inclusion $\iota^{\text{inv}} : \text{Blpr}_{\mathbb{F}_{1^2}} \to \text{OBlpr}$ as a subcategory. We also write B^{inv} for $B \otimes_{\mathbb{F}_1} \mathbb{F}_{1^2}$.

Lemma 2.4. *Every ordered blueprint with -1 is an algebraic blueprint.*

Proof. By multiplication with -1, a relation $\sum a_i \leqslant \sum b_j$ implies that $\sum -a_i \leqslant \sum -b_j$ and thus

$$\sum b_j \quad\equiv\quad \sum b_j + \sum -a_i + \sum a_i \quad\leqslant\quad \sum b_j + \sum -b_j + \sum a_i \quad\equiv\quad \sum a_i,$$

which shows that, indeed, $\sum a_i \equiv \sum b_j$. □

Summing up all facts, we justified the notation $\mathrm{Blpr}^{\mathrm{inv}}$ or $\mathrm{Blpr}_{\mathbb{F}_{1^2}}$ for the full subcategory of blueprints with -1 in OBlpr.

2.5. The universal ordered semiring. In this text, an *ordered semiring* is a semiring R together with a partial order $\leqslant$ that satisfies for all $x,y,z,t \in R$

(S1) $x \leqslant y$ and $z \leqslant t$ implies $x+z \leqslant y+t$; *(additive)*
(S2) $x \leqslant y$ and $z \leqslant t$ implies $xz \leqslant yt$. *(multiplicative)*

A morphism of ordered semirings is an order-preserving homomorphism of ordered semirings that maps 0 to 0 and 1 to 1.

Let $B = A /\!\!/ \mathcal{R}$ be an ordered blueprint. The *universal ordered semiring B^+ associated with B* is the semiring $\mathbb{N}[A]^+/\mathcal{R}^{\mathrm{core}}$ together with the partial order defined by

$$\big[\sum a_i\big] \leqslant \big[\sum b_j\big] \quad \text{in } B^+ \text{ if and only if} \quad \sum a_i \leqslant \sum b_j \quad \text{in } B.$$

Note that B^+ is well-defined as an ordered semiring: by additivity and multiplicativity of $\mathcal{R}$, B^+ inherits the structure of a semiring as a quotient of the semiring $\mathbb{N}[A]^+$; by transitivity, the partial order on B^+ is well-defined (as a relation on B^+); by reflexivity and transitivity of $\mathcal{R}$ and the definition of B^+, this relation is indeed a partial order on B^+; again by additivity and multiplicativity, the partial order of B^+ is additive and multiplicative.

A morphism $f : B \to C$ of ordered blueprints induces a morphism of ordered semirings $f^+ : B^+ \to C^+$ that is defined by $f([\sum a_i]) = [\sum f(a_i)]$. This establishes the functor $(-)^+$ from OBlpr to the category of ordered semirings.

Conversely, we can consider every ordered semiring $(R,\leqslant)$ as an ordered blueprint $B = A /\!\!/ \mathcal{R}$: we let A be the underlying multiplicative monoid of R and define

$$\mathcal{R} \quad=\quad \{ \sum a_i \leqslant \sum b_j \mid \sum a_i \leqslant \sum b_j \text{ in } R \}.$$

Then a map between ordered semirings is the same as a morphism between the associated ordered blueprints. This defines an embedding ι^+ of the category of ordered semirings as a full subcategory of the category of ordered blueprints, which is a left adjoint to $(-)^+$. Moreover, $(-)^+ \circ \iota^+$ is isomorphic to the identity functor on the category of ordered semirings.

From now on, we identify the category of ordered semirings with the essential image of ι^+, which allows us to talk about morphisms from ordered blueprints into ordered semirings. We see that an ordered blueprint $B = A /\!\!/ \mathcal{R}$ comes with the morphism $B \to B^+$ that sends $a \in A$ to the class $[a] \in B^+$. This morphism is universal for morphisms from B into ordered semirings.

We identify the category of semirings with the subcategory of trivially ordered semirings. This coincides with the realization of semirings as algebraic blueprints, followed by the embedding of Blpr into OBlpr. We derive yet another characterization of algebraic blueprints in OBlpr.

Lemma 2.5. *An ordered blueprint B is an algebraic blueprint if and only if B^+ is trivially ordered.* □

Remark 2.6. The subaddition $\mathcal{R}$ of B can be recovered from the embedding $B^\bullet \subset B^+$ as

$$\mathcal{R} \quad=\quad \Big\langle \sum a_i \leqslant \sum b_j \,\Big|\, \sum a_i \leqslant \sum b_j \text{ in } B^+ \Big\rangle.$$

An order preserving homomorphism $f : B_1^+ \to B_2^+$ of ordered semirings comes from a morphism of blueprints $B_1 \to B_2$ if and only if f maps the underlying monoid of B_1 to the underlying monoid of B_2.

This yields an equivalence between the category of ordered blueprints, as defined in this section, with the category of inclusions $B^\bullet \subset B^+$ as considered in the introduction.

2.6. Monomial blueprints. The first step towards realizing norms and valuations as morphisms is to concentrate on inequalities of the form $a \leqslant \sum b_j$. We can associate with every ordered blueprint an ordered blueprint based on inequalities of this sort in a functorial way.

A *(left) monomial relation* is a relation of the form $a \leqslant \sum b_j$. A *(left) monomial (ordered) blueprint* is an ordered blueprint B whose subaddition $\mathcal{R}$ is generated by monomial relations, i.e.

$$\mathcal{R} \quad = \quad \big\langle\, a \leqslant \sum b_j \;\big|\; (a, \sum b_j) \in \mathcal{R} \,\big\rangle .$$

We denote the full subcategory of monomial blueprints in OBlpr by $\mathrm{OBlpr}^{\mathrm{mon}}$.

Let $B = A /\!\!/ \mathcal{R}$ be an ordered blueprint. The *associated monomial blueprint* is defined as $B = A /\!\!/ \mathcal{R}^{\mathrm{mon}}$ with

$$\mathcal{R}^{\mathrm{mon}} \quad = \quad \big\langle\, a \leqslant \sum b_j \;\big|\; a \leqslant \sum b_j \text{ in } B \,\big\rangle .$$

The obvious inclusion $B^{\mathrm{mon}} \to B$ is universal for all morphisms from a monomial blueprint to B, and it is an isomorphism if and only if B itself is monomial. This defines a right adjoint and left inverse $(-)^{\mathrm{mon}} : \mathrm{OBlpr} \to \mathrm{OBlpr}^{\mathrm{mon}}$ to the inclusion functor $\iota^{\mathrm{mon}} : \mathrm{OBlpr}^{\mathrm{mon}} \to \mathrm{OBlpr}$.

Remark 2.7. Paugam's category Halos of halos and halo morphisms in [44] appears naturally as a full subcategory of $\mathrm{OBlpr}^{\mathrm{mon}}$. A *halo* is an ordered semiring and a *(multiplicative) halo morphism* is a order preserving multiplicative map $f : B_1 \to B_2$ of ordered semirings such that $f(0) = 0$, $f(1) = 1$ and $f(a+b) \leqslant f(a) + f(b)$. If we consider B_1 and B_2 as ordered blueprints, then it is easily seen that a map $f : B_1 \to B_2$ is a halo morphism if and only if the composition $f' : B_1^{\mathrm{mon}} \to B_1 \to B_2$ is a morphism of ordered blueprints. By the universal property of a monomial blueprint, f' factors uniquely through the morphism $f^{\mathrm{mon}} : B_1^{\mathrm{mon}} \to B_2^{\mathrm{mon}}$ of blueprints. This defines a fully faithful embedding $(-)^{\mathrm{mon}} : \mathrm{Halos} \to \mathrm{OBlpr}^{\mathrm{mon}}$.

Remark 2.8. Another closely related concept is the notion of a hyperring, cf. [49], [28], [55], [56] and [13]. A *(commutative) hyperring* is a multiplicative monoid R together with a function $f : R \times R \to \mathcal{P}(R)$ into the power set $\mathcal{P}(R)$ of R that associates defines the sum $a+b$ of two elements $a, b \in R$ as a non-empty subset of R. This function satisfies certain axioms in analogy to the classical ring axioms. Given a hyperring R, we define the blueprint $B = R^\bullet /\!\!/ \mathcal{R}_R$ where $\mathcal{R}_R$ is generated by the monomial relations $c \leqslant a+b$ whenever $c \in a+b$. This defines a full embedding of the category of hyperrings into the category of monomial blueprints.

2.7. Partially additive blueprints. A *partially additive blueprint* is an algebraic blueprint B whose preaddition is generated by relations of the form $a \equiv \sum b_j$. We denote the full subcategory of partially additive blueprints in OBlpr by $\mathrm{Blpr}^{\mathrm{padd}}$. To an ordered blueprint $B = A /\!\!/ \mathcal{R}$, we can associate a partially additive blueprint $B^{\mathrm{padd}} = B /\!\!/ \mathcal{R}^{\mathrm{padd}}$ where $\mathcal{R}^{\mathrm{padd}}$ is generated by all relations $a \equiv \sum b_j$ that are contained in $\mathcal{R}$. The inclusion $B^{\mathrm{padd}} \to B$ is universal for all morphism from a partially ordered blueprint to B. This defines a right adjoint and left inverse $(-)^{\mathrm{padd}} : \mathrm{OBlpr} \to \mathrm{Blpr}^{\mathrm{padd}}$ to the inclusion functor $\iota^{\mathrm{padd}} : \mathrm{Blpr}^{\mathrm{padd}} \to \mathrm{OBlpr}$.

By the very definition of partially additive and monomial blueprints, we obtain that $(-)^{\mathrm{mon}} : \mathrm{Blpr}^{\mathrm{padd}} \to \mathrm{Blpr}^{\mathrm{mon}}$ is a fully faithful embedding of categories with left adjoint and left inverse $(-)^{\mathrm{hull}} : \mathrm{Blpr}^{\mathrm{mon}} \to \mathrm{Blpr}^{\mathrm{padd}}$. This means that we model the category of partially additive blueprints as monomial blueprints, which will be of importance for our observations in section 7.2. Note that examples of partially additive blueprints include semirings, monoids and blueprints with -1.

Remark 2.9. Partially additive blueprints are closely connected to the idea of a sesquiad, cf. [15]. Namely, a *sesquiad* is a monoid A together with partial functions $f_n : A^n \dashrightarrow A$ (for $n \geqslant 1$)

that satisfies certain axioms. The functions f_n express the sum $a = \sum b_j \in A$ if $(b_1,\dots,b_n)$ is in the domain of f_n. Equivalently, a sesquiad is a partially additive blueprint $B = A/\!\!/\mathcal{R}$ that is *cancellative*, i.e. the canonical morphism $B^+ \to B^{+,\mathrm{inv}}$ is injective.

2.8. Totally positive blueprints. A *totally positive* blueprint is an ordered blueprint B that satisfies $0 \leqslant 1$. We denote the full subcategory of totally positive blueprints in OBlpr by $\mathrm{OBlpr}^{\mathrm{pos}}$.

Let $B = A/\!\!/\mathcal{R}$ be an ordered blueprint. The *associated totally positive blueprint* is defined as $B^{\mathrm{pos}} = B/\!\!/\langle 0 \leqslant 1\rangle$, which is the same as $B \otimes_{\mathbb{F}_1} \big(\mathbb{F}_1/\!\!/\langle 0 \leqslant 1\rangle\big)$. The obvious morphism $B \to B^{\mathrm{pos}}$ is universal for all morphisms from B to a totally positive blueprint, and it is an isomorphism if and only if B itself is totally positive. This defines the functor $(-)^{\mathrm{pos}} : \mathrm{OBlpr} \to \mathrm{OBlpr}^{\mathrm{pos}}$, which is left adjoint and left inverse to the inclusion $\mathrm{OBlpr}^{\mathrm{pos}} \to \mathrm{OBlpr}$ as a subcategory.

Lemma 2.10. *Let B be an ordered blueprint. Then the following are equivalent.*

(i) *B is totally positive;*
(ii) *$0 \leqslant a$ for all $a \in B$;*
(iii) *$\sum a_i + \sum c_k \leqslant \sum b_j$ implies $\sum a_i \leqslant \sum b_j$.*

Proof. Let B satisfy (i). Multiplying the relation $0 \leqslant 1$ by $a \in B$ yields (ii).

Let B satisfy (ii). A relation $\sum a_i + \sum c_k \leqslant \sum b_j$ implies $\sum a_i \equiv \sum a_i + \sum 0 \leqslant \sum a_i + \sum c_k \leqslant \sum b_j$, which is (iii).

Let B satisfy (iii). Then $0+1 \leqslant 1$ implies $0 \leqslant 1$. Thus (i). □

Corollary 2.11. *Let B be an ordered blueprint.*

(i) *If $a \leqslant 0$ in B, then $a \equiv 0$ in B^{pos}.*
(ii) *If $1 + \sum c_k \leqslant 0$ for some c_k in B, then B^{pos} is the trivial blueprint. Thus if B is with -1, then $B^{\mathrm{pos}} = 0$.*
(iii) *The canonical morphism $B \to B^{\mathrm{pos}}$ is the identity between the respective underlying monoids if and only if*

$$a + \sum c_k \leqslant b \quad \text{and} \quad b + \sum d_l \leqslant a \quad \text{imply} \quad a = b \quad \text{in} \quad B.$$

If B is a semiring, then this is the case if and only if $a + c + d = a$ implies $a + c = a$.

Proof. By Lemma 2.10 (ii), we have $0 \leqslant a$ for all a in B^{pos}. If $a \leqslant 0$ in B, then $a \equiv 0$ in B^{pos}, which shows (i).

By Lemma 2.10 (iii), a relation $1 + \sum c_k \leqslant 0$ in B implies $1 \leqslant 0$ in B^{pos}. Thus $0 \equiv 1$ by (i), which is equivalent with $B^{\mathrm{pos}} = 0$. This shows (ii).

We prove (iii). By definition, the canonical morphism $B \to B^{\mathrm{pos}}$ is a surjective monoid morphism. Two relations $a + \sum c_k \leqslant b$ and $b + \sum d_l \leqslant a$ in B imply that $a \equiv b$ in B^{pos}, which shows that already $a \equiv b$ in B if $B \to B^{\mathrm{pos}}$ is an isomorphism between the respective underlying monoids.

Since all additional relations in B^{pos} are generated by leaving out summands on the left hand side of relations in B, we get only new relations $a \leqslant b$ in B^{pos} for relations of the form $a + \sum c_k \leqslant b$ in B. From this, the other direction of the claim follows.

If B is a semiring, we can substitute $\sum c_k$ by its sum c and $\sum d_l$ by its sum d. Moreover, the inequalities $a + c \leqslant b$ and $b + d \leqslant a$ are equalities, which yields

$$a + c + d \quad = \quad b + d \quad = \quad a.$$

On the other hand, $a + c + d = a$ yields $b + d = a$ if we set $b = a + c$, i.e. we re-obtain the two equations that we started with. With the same substitution $b = a + c$, the equation $a = b$ is equivalent to $a = a + c$. This proves the latter claim of (iii). □

2.9. Strictly conic blueprints. In this section, we encounter the question under which conditions $(B^{\text{pos}})^{\text{core}}$ is isomorphic to B.

A *strictly conic ordered blueprint* is an ordered blueprint B that satisfies

(B8) $\sum a_i + \sum c_k \leqslant \sum b_j$ and $\sum b_j + \sum d_l \leqslant \sum a_i$ imply $\sum a_i \equiv \sum b_j$. *(strictly conic)*

We denote the full subcategory of strictly conic ordered blueprints in OBlpr by $\text{OBlpr}^{\text{conic}}$. Let $B = A /\!\!/ \mathcal{R}$ be an ordered blueprint. The *strictly conic ordered blueprint associated with B* is the ordered blueprint $B^{\text{conic}} = A /\!\!/ \mathcal{R}^{\text{conic}}$ where $\mathcal{R}^{\text{conic}}$ is generated by $\mathcal{R}$ and

$$\Big\{ \sum a_i \equiv \sum b_j \;\Big|\; \sum a_i + \sum c_k \leqslant \sum b_j \text{ and } \sum b_j + \sum d_l \leqslant \sum a_i \Big\}.$$

The associated strictly conic ordered blueprint comes together with the obvious morphism $B \to B^{\text{conic}}$, which is universal for all morphisms from B to a strictly conic ordered blueprint. This defines the left adjoint and left inverse $(-)^{\text{conic}} : \text{OBlpr} \to \text{OBlpr}^{\text{conic}}$ to the inclusion $\text{OBlpr}^{\text{conic}} \to \text{OBlpr}$ as a subcategory.

In order to investigate the relation between an ordered blueprint B and $(B^{\text{pos}})^{\text{core}}$, consider the commutative diagram

$$\begin{tikzcd}
 & B \arrow[dr, "\alpha_B^{\text{pos}}"] & \\
B^{\text{core}} \arrow[ur, "\alpha_B^{\text{core}}"] \arrow[dr, "\beta_B = (\alpha_B^{\text{pos}})^{\text{core}}"'] & & B^{\text{pos}}. \\
 & (B^{\text{pos}})^{\text{core}} \arrow[ur, "\alpha_{B^{\text{pos}}}^{\text{core}}"'] &
\end{tikzcd}$$

Proposition 2.12. *The map β_B is an isomorphism of blueprints if and only if B is strictly conic.*

Proof. Assume that β_B is an isomorphism. Since $\sum a_i + \sum c_k \leqslant \sum b_j$ and $\sum b_j + \sum d_l \leqslant \sum a_i$ in B imply $\sum a_i \equiv \sum b_j$ in B^{pos}, and therefore in $(B^{\text{pos}})^{\text{core}}$, this must also hold in B^{core} as β_B is an isomorphism. By the definition of the algebraic core, $\sum a_i \equiv \sum b_j$ in B, which shows that B is strictly conic.

To prove the reverse direction, assume that B is strictly conic. By Corollary 2.11 (iii), α_B^{pos} is an isomorphism between the underlying monoids. The maps α_B^{core} and $\alpha_{B^{\text{pos}}}^{\text{core}}$ are so, too, by the definition of the algebraic core. This shows that β_B is an isomorphism between the underlying monoids.

Given an equality $\sum a_i \equiv \sum b_j$ in $(B^{\text{pos}})^{\text{core}}$, this must already hold in B^{pos}. By the definition of B^{pos}, there must be relations of the form $\sum a_i + \sum c_k \leqslant \sum b_j$ and $\sum b_j + \sum d_l \leqslant \sum a_i$ in B. As B is strictly conic, we have $\sum a_i \equiv \sum b_j$ in B and therefore in B^{core}. This shows that β_B is an isomorphism. □

Let $\text{Blpr}^{\text{conic}}$ be the full subcategory of strictly conic algebraic blueprints in OBlpr.

Corollary 2.13. *Let B be an algebraic blueprint. Then $(B^{\text{pos}})^{\text{core}}$ is isomorphic to B if and only if B is strictly conic. Consequently, $(-)^{\text{pos}}$ embeds $\text{Blpr}^{\text{conic}}$ fully faithfully into $\text{OBlpr}^{\text{pos}}$, with left-inverse $(-)^{\text{core}}$.* □

Corollary 2.14. *Let B be a semiring. Then the following are equivalent.*

(i) *B is strictly conic.*
(ii) *$B = B^{\text{core}} \to (B^{\text{pos}})^{\text{core}}$ is an isomorphism.*
(iii) *$a + c + d = a$ implies $a + c = a$.*
(iv) *$B \to B^{\text{pos}}$ is an isomorphism between the underlying monoids.*

Proof. The equivalence of (i) and (ii) is Corollary 2.13. The equivalence of (iii) and (iv) is Corollary 2.11 (iii). That (iii) is equivalent to (i) is shown analogous to the proof of Corollary 2.11 (iii). □

Recall that a *strict semiring* is a semiring B such that an equality $a+b=0$ implies $a=0$. An *idempotent blueprint* is a blueprint B with $1+1\equiv 1$, which implies $a+a\equiv a$ for every $a\in B$. We denote the full subcategory of OBlpr of idempotent blueprints by $\mathrm{OBlpr}^{\mathrm{idem}}$. Its initial object is the boolean semiring $\mathbb{B}=\{0,1\}/\!\!/\langle 1+1\equiv 1\rangle$ and the functor $-\otimes_{\mathbb{F}_1}\mathbb{B}$ is a left adjoint and left inverse to the inclusion functor $\mathrm{OBlpr}^{\mathrm{idem}}\to\mathrm{OBlpr}$. A *nonnegative blueprint* is a blueprint B such that the only element $a\in B$ with $a\leqslant 0$ is $a=0$.

Lemma 2.15. *The following holds true.*

- (i) *A strictly conic semiring is strict.*
- (ii) *An idempotent blueprint is strictly conic.*
- (iii) *A totally positive blueprint is strictly conic.*
- (iv) *A nonnegative monomial blueprint is strictly conic.*

Proof. Let B be a strictly conic semiring. Since $a+b=0$ implies $0+a+b=0$ and thus $a=0+a=0$, B is a strict semiring. Thus (i).

Let B be an idempotent blueprint and assume that $\sum a_i+\sum c_k\equiv\sum b_j$ and $\sum b_j+\sum d_l\equiv\sum a_i$. Then

$$\begin{aligned}\sum a_i \;\equiv\; \sum a_i+\sum a_i \;\equiv\; \sum a_i+\sum b_j+\sum d_l \;&\equiv\; \sum a_i+\sum b_j+\sum b_j+\sum d_l\\ &\equiv\; \sum a_i+\sum a_i+\sum c_k+\sum b_j+\sum d_l\\ &\equiv\; \sum a_i+\sum b_j+\sum c_k+\sum d_l,\end{aligned}$$

which, by reasons of symmetry, equals $\sum b_j$. Therefore B is strictly conic. Thus (ii).

If B is totally positive, then the relations $\sum a_i+\sum c_k\equiv\sum b_j$ and $\sum b_j+\sum d_l\equiv\sum a_i$ imply $\sum a_i\leqslant\sum b_j$ and $\sum b_j\leqslant\sum a_i$. Thus $\sum a_i\equiv\sum b_j$ as desired. This shows (iii).

Let B be non-negative and monomial and consider $\sum a_i+\sum c_k\leqslant\sum b_j$ and $\sum b_j+\sum d_l\leqslant\sum a_i$ where we assume that a_i,b_j,c_k,d_l are non-zero. These relations are generated by left monomial relations of the form $a'\leqslant\sum b'_j$, which contain at least one nonzero term b'_j if a' is nonzero since B is nonnegative. Therefore $\#\{a_i,c_k\}\leqslant\#\{b_j\}$ and $\#\{b_j,d_l\}\leqslant\#\{a_i\}$, which is only possible if $\{c_k\}=\{d_l\}=\emptyset$. Consequently, $\sum a_i\equiv\sum b_j$, which shows that B is strictly conic as claimed in (iv). □

Corollary 2.16. *If B is an idempotent ordered blueprint, then the canonical morphism $B^{\mathrm{core}}\to(B^{\mathrm{pos}})^{\mathrm{core}}$ is an isomorphism and the canonical morphism $B\to B^{\mathrm{pos}}$ is a bijection.*

Proof. By Lemma 2.15, B is strictly conic and by Proposition 2.12, $B^{\mathrm{core}}\to(B^{\mathrm{pos}})^{\mathrm{core}}$ is an isomorphism. Consequently, we obtain a bijection $B^{\mathrm{core}}=(B^{\mathrm{pos}})^{\mathrm{core}}\to B^{\mathrm{pos}}$, which factors into the canonical morphisms $B^{\mathrm{core}}\to B$ and $B\to B^{\mathrm{pos}}$. Since $B^{\mathrm{core}}\to B$ is a bijection, we conclude that $B\to B^{\mathrm{pos}}$ is also a bijection. □

Example 2.17 (A strict semiring that is not strictly conic)**.** The semiring $R=\mathbb{N}[S,T]^+/\!\!/\langle 1+S+T\equiv 1\rangle$ is obviously a strict semiring. However, $1+S+T=1$ while $1+S\neq 1$, which shows that R is not strictly conic.

Example 2.18 (Idempotent semirings)**.** We can endow an idempotent semiring B with the partial order $a\leqslant b$ if and only if there is a $c\in B$ such that $a+c\equiv b$. By Corollary 2.16, $B\to B^{\mathrm{pos}}$ is a bijection and by Lemma 2.10 (iii), the partial order associated with B is equal to the restriction of the subaddition of B^{pos} to its underlying set, which is equal to B. This observation is crucial for our reinterpretation of valuations in idempotent semirings in terms of morphisms into the associated totally positive blueprint.

Example 2.19 (Non-negative reals and tropical numbers)**.** The non-negative real numbers $\mathbb{R}_{\geqslant 0}$ form a strictly conic semiring with respect to to their usual multiplication and addition. A more

general class of strictly conic semirings are the non-negative reals together with the modified addition

$$a +^t b \quad = \quad \begin{cases} (a^t + b^t)^{1/t} & \text{if } t \in [1, \infty) \\ \max\{a, b\} & \text{if } t = \infty. \end{cases}$$

We denote this semiring by $\mathbb{R}^t_{\geqslant 0}$, and $\mathbb{R}^\infty_{\geqslant 0}$ by $\mathbb{T}$, the *tropical numbers*. Note that taking logarithms identifies $\mathbb{T}$ with the max/$+$-semiring $\mathbb{R} \cup \{-\infty\}$, which is more commonly considered as the semiring of tropical numbers.

By Corollary 2.14, $\mathbb{R}_{\geqslant 0} \to \mathbb{R}^{\text{pos}}_{\geqslant 0}$ and $\mathbb{T} \to \mathbb{T}^{\text{pos}}$ are isomorphisms between the underlying monoids, which is in both cases $\mathbb{R}^\bullet_{\geqslant 0}$. The subaddition of $\mathbb{R}^{\text{pos}}_{\geqslant 0}$ is

$$\left\{ \sum a_i \leqslant \sum b_j \;\middle|\; \sum a_i + c \equiv \sum b_j \text{ for some } c \text{ in } \mathbb{R}_{\geqslant 0} \right\},$$

which coincides with the natural order of $\mathbb{R}_{\geqslant 0}$ if we identify $\sum a_i$ with its sum in $\mathbb{R}_{\geqslant 0}$. The subaddition of $\mathbb{T}^{\text{pos}}$ is

$$\left\{ \sum a_i \leqslant \sum b_j \;\middle|\; \max\{a_i\} \leqslant \max\{b_j\} \text{ in } \mathbb{R}_{\geqslant 0} \right\},$$

which also induces the natural order of $\mathbb{R}_{\geqslant 0}$ if restricted to relations of the form $a \leqslant b$ with $a, b \in \mathbb{R}_{\geqslant 0}$. For the consideration of non-archimedean norms, it is useful to observe the following comparison between these two totally positive blueprints.

Lemma 2.20. *The identity between the underlying monoids induces a morphism*

$$(\mathbb{T}^{\text{pos}})^{\text{mon}} \longrightarrow (\mathbb{R}^{\text{pos}}_{\geqslant 0})^{\text{mon}}$$

of monomial blueprints.

Proof. By the above description of the respective preadditions of $\mathbb{R}^{\text{pos}}_{\geqslant 0}$ and $\mathbb{T}^{\text{pos}}$, we conclude that the preaddition of $(\mathbb{T}^{\text{pos}})^{\text{mon}}$ is generated by the relations of the form $a \leqslant \sum b_j$ whenever $a \leqslant \max\{b_j\}$ as elements of $\mathbb{R}_{\geqslant 0}$. These relations are also contained in $(\mathbb{R}^{\text{pos}}_{\geqslant 0})^{\text{mon}}$. This shows that the identity map $(\mathbb{T}^{\text{pos}})^{\text{mon}} \to (\mathbb{R}^{\text{pos}}_{\geqslant 0})^{\text{mon}}$ between the underlying monoids is a morphism of blueprints. $\square$

2.10. Overview of subcategories. We denote the category of semirings by SRings and the category of rings by Rings. They both form full subcategories of OBlpr by associating with a (semi)ring R the blueprint $B = R^\bullet /\!\!/ \langle \mathcal{R} \rangle$ where $\mathcal{R} = \{\sum a_i \equiv \sum b_j | \sum a_i = \sum b_j \text{ in } R\}$. Note that a semiring is a ring if and only if it is with inverses.

Note also the following facts: a monomial blueprint that is algebraic is a monoid; a totally positive algebraic blueprint is trivial; a strictly conic blueprint with inverses is trivial.

Using the previous results on the relations of the different subcategories of OBlpr, we can illustrate the subcategories of OBlpr that are relevant to this text as in Figure 1. An inclusion of areas indicates an inclusion of subcategories, and areas with empty intersection indicates that the only common object in the corresponding subcategories is $\{0\}$.

3. Valuations

With the formalism developed in the previous section, we are ready to give the general definition of a valuation, which restricts to the different concepts of (semi)norms and valuations in particular cases.

Definition 3.1. Let B and S be two ordered blueprints. A *valuation of B in S* is a morphism $v^\bullet : B^\bullet \to S^\bullet$ between the underlying monoids that admits a morphism $\tilde{v} : B^{\text{mon}} \to S^{\text{pos}}$ such that the diagram

$$\begin{tikzcd} B^\bullet \arrow[r, "v^\bullet"] \arrow[d] & S^\bullet \arrow[d] \\ B^{\text{mon}} \arrow[r, "\tilde{v}"] & S^{\text{pos}} \end{tikzcd}$$

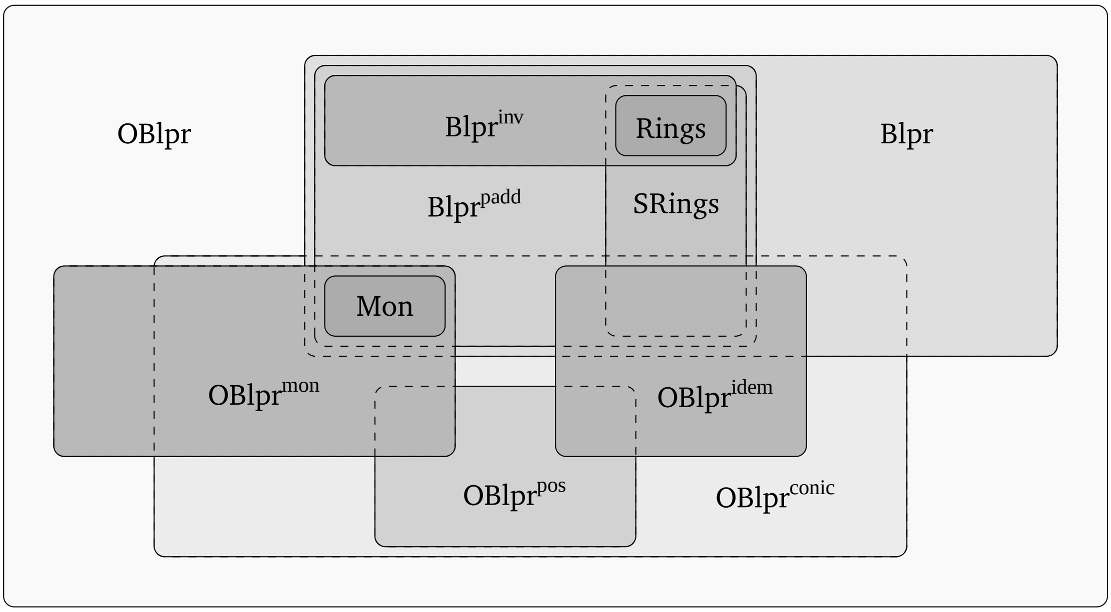

FIGURE 1. Some relevant subcategories of OBlpr

commutes. We write $v : B \to S$ for a valuation v of B in S.

Since the canonical morphism $S^\bullet \to S^{\mathrm{mon}}$ is a bijection, $\tilde{v}$ is uniquely determined by v. In other words, a valuation is a multiplicative map $v : B \to S$ such that the composition map

$$\tilde{v} : \; B^{\mathrm{mon}} \longrightarrow B \xrightarrow{\;v\;} S \longrightarrow S^{\mathrm{pos}}$$

is a morphism of ordered blueprints.

Note that v is uniquely determined by $\tilde{v}$ if $S \to S^{\mathrm{pos}}$ is a bijection. By Lemma 2.10 (iii), this holds for strictly conic S, which is the case that we are interested most in this paper.

Note further that every morphism $v : B \to S$ is a valuation since the diagram

$$\begin{tikzcd} B^\bullet \arrow[rrr, "v^\bullet"] \arrow[d] & & & S^\bullet \arrow[d] \\ B^{\mathrm{mon}} \arrow[r] \arrow[rrr, bend left, "\tilde{v}"] & B \arrow[r, "v"] & S \arrow[r] & S^{\mathrm{pos}} \end{tikzcd}$$

commutes. If B is monomial and S totally positive, then every valuation $v : B \to S$ is a morphism.

3.1. Seminorms. Let R be a ring. A *seminorm* on R is a monoid morphism $v : R \to \mathbb{R}_{\geqslant 0}$ that satisfies the *triangle inequality* $v(a+b) \leqslant v(a)+v(b)$ for all $a,b \in R$. A *non-archimedean seminorm* is a monoid morphism that satisfies the *strong triangle inequality* $v(a+b) \leqslant \max\{v(a),v(b)\}$.

Lemma 3.2. *A map $v : R \to \mathbb{R}_{\geqslant 0}$ is a seminorm if and only if the composition*

$$\tilde{v} : \; R^{\mathrm{mon}} \longrightarrow R \xrightarrow{\;v\;} \mathbb{R}_{\geqslant 0} \longrightarrow \mathbb{R}_{\geqslant 0}^{\mathrm{pos}}$$

is a morphism of ordered blueprints. A map $v : R \to \mathbb{R}_{\geqslant 0}$ is a non-archimedean seminorm if and only if the composition

$$\tilde{v} : \; R^{\mathrm{mon}} \longrightarrow R \xrightarrow{\;v\;} \mathbb{T} \longrightarrow \mathbb{T}^{\mathrm{pos}}$$

is a morphism of ordered blueprints where we identify $\mathbb{R}_{\geqslant 0}$ as a set with the tropical semiring $\mathbb{T}$.

Proof. Since the canonical maps $R^{\text{mon}} \to R$, $\mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}^{\text{pos}}$ and $\mathbb{T} \to \mathbb{T}^{\text{pos}}$ are bijections, it is clear that the map $v: R \to \mathbb{R}_{\geqslant 0}$ is a morphism of monoids if and only if the composition $R^{\text{mon}} \to \mathbb{R}_{\geqslant 0}^{\text{pos}}$ or $R^{\text{mon}} \to \mathbb{T}^{\text{pos}}$, respectively, is a monoid morphism.

A monoid morphism $v: R \to \mathbb{R}_{\geqslant 0}$ is a seminorm if and only if it satisfies $v(b) \leqslant \sum v(a_i)$ for arbitrary sums $b \equiv \sum a_i$. The subaddition of R^{mon} is generated by the relations $b \leqslant \sum a_i$ for which $b \equiv \sum a_i$ in R. Such a relation is mapped to the relation $\tilde{v}(b) \leqslant \sum \tilde{v}(a_i)$, which is in the subaddition of $\mathbb{R}_{\geqslant 0}^{\text{pos}}$ by the triangle inequality for v and Example 2.19. This means that $\tilde{v}$ is a morphism.

Assume, conversely, that $\tilde{v}: R^{\text{mon}} \to \mathbb{R}_{\geqslant 0}^{\text{pos}}$ is a morphism of ordered blueprints and consider $b \equiv \sum a_i$ in R. Then we have $b \leqslant \sum a_i$ in R^{mon} and $\tilde{v}(b) \leqslant \sum \tilde{v}(a_i)$ in $\mathbb{R}_{\geqslant 0}^{\text{pos}}$, which means that $v(b) \leqslant \sum v(a_i)$ with respect to the natural order of $\mathbb{R}_{\geqslant 0}$. This means that $\tilde{v}$ is a seminorm.

Since the addition of the tropical semiring $\mathbb{T}$ is $a+b = \max\{a,b\}$, the latter claim of the lemma follows by the same argument as the former one. □

Remark 3.3. The non-archimedean seminorms can be characterized as the following seminorms. By the universal property of a monomial blueprint, a morphism $R^{\text{mon}} \to \mathbb{R}_{\geqslant 0}^{\text{pos}}$ factorizes uniquely into $R^{\text{mon}} \to (\mathbb{R}_{\geqslant 0}^{\text{pos}})^{\text{mon}} \to \mathbb{R}_{\geqslant 0}^{\text{pos}}$. Using the morphism $(\mathbb{T}^{\text{pos}})^{\text{mon}} \to (\mathbb{R}_{\geqslant 0}^{\text{pos}})^{\text{mon}}$ from Lemma 2.20, we see that the seminorm $v: R \to \mathbb{R}_{\geqslant 0}$ is *non-archimedean* if and only if $R^{\text{mon}} \to \mathbb{R}_{\geqslant 0}^{\text{pos}}$ factors into

$$R^{\text{mon}} \longrightarrow (\mathbb{T}^{\text{pos}})^{\text{mon}} \longrightarrow (\mathbb{R}_{\geqslant 0}^{\text{pos}})^{\text{mon}} \longrightarrow \mathbb{R}_{\geqslant 0}^{\text{pos}}.$$

3.2. Krull valuations.

Let Γ be an multiplicatively written partially ordered commutative semigroup with unit 1. We denote by Γ_0 the ordered blueprint $(\Gamma \cup \{0\}, \mathcal{R})$ where $\mathcal{R}$ is generated by the partial order of Γ and the relation $0 \leqslant 1$.

Proposition 3.4. *The tensor product $\Gamma_{\mathbb{B}} = \Gamma_0 \otimes_{\mathbb{F}_1} \mathbb{B}$ is a totally positive blueprint with idempotent algebraic core $\Gamma_{\mathbb{B}}^{\text{core}} = (\Gamma_{\mathbb{B}})^{\text{core}}$. The canonical morphism $\Gamma_0 \to \Gamma_{\mathbb{B}}$ is bijective and the canonical morphism $(\Gamma_{\mathbb{B}}^{\text{core}})^{\text{pos}} \to \Gamma_{\mathbb{B}}$ is an isomorphism. If Γ is totally ordered, then $\Gamma_{\mathbb{B}}^{\text{core}}$ is a semiring with $a+b = \max\{a,b\}$.*

Proof. By Corollary 2.11 (i), $\Gamma_{\mathbb{B}}$ is totally positive, and by definition, the algebraic core of $\Gamma_{\mathbb{B}} = \Gamma_0 /\!\!/ \langle 1+1 \equiv 1\rangle$ is idempotent.

Since $\Gamma_{\mathbb{B}} = \Gamma_0 /\!\!/ \langle 1+1 \equiv 1\rangle$, it is clear that $\Gamma_0 \to \Gamma_{\mathbb{B}}$ is surjective. Since the core of Γ_0 has the trivial preaddition $\langle \emptyset \rangle$, $\Gamma_0 \to \Gamma_{\mathbb{B}}$ is injective.

Let $a \leqslant b$ be a relation in Γ_0. Then we have $b \equiv 0+b \leqslant a+b \leqslant b+b \equiv b$ in $\Gamma_{\mathbb{B}}$ and $a+b \equiv b$ in its algebraic core B. Consequently, $a \leqslant b$ in B^{pos}. Since also the relation $1+1 \equiv 1$ is in B and B^{pos} and the subaddition of $\Gamma_{\mathbb{B}}$ is generated by relations of the form $a \leqslant b$ and $1+1 \equiv 1$, the canonical morphism $B^{\text{pos}} \to \Gamma_{\mathbb{B}}$ is an isomorphism.

If Γ is totally ordered, then for all a and b, the sum $a+b \equiv \max\{a,b\}$ is defined by the above argument. Therefore B is a semiring. □

Let k be a field and Γ a (multiplicatively written) totally ordered group. A *Krull valuation of k with value group Γ* is a surjective monoid map $v: k \to \Gamma_0$ with $v(a+b) \leqslant \max\{v(a), v(b)\}$.

Corollary 3.5. *A surjective map $v: k \to \Gamma$ is a Krull valuation if and only if the composition*

$$\tilde{v}: \ k^{\text{mon}} \longrightarrow k \xrightarrow{\ v\ } \Gamma_0 \longrightarrow \Gamma_{\mathbb{B}}$$

is a morphism of ordered blueprints.

Proof. Since $k^{\text{mon}} \to k$ and $\Gamma_0 \to \Gamma_{\mathbb{B}}$ are bijections, cf. Proposition 3.4, v is a monoid morphism if and only if $\tilde{v}$ is so. By Proposition 3.4, the addition of the semiring $\Gamma_{\mathbb{B}}^{\text{core}}$ is defined as $a+b = \max\{a,b\}$ (with respect to the order of Γ). Therefore the same arguments as in the

proof of Lemma 3.2 show that v satisfies the strong triangle inequality if and only if $\tilde{v}$ maps relations $b \leqslant \sum a_i$ in k^{mon} to relations $\tilde{v}(b) \leqslant \sum \tilde{v}(a_i)$ in $\Gamma_{\mathbb{B}}$. $\square$

Remark 3.6. Note that usually, the group Γ is written additively and considered with the reversed order, i.e. we have $v(a+b) \leqslant \min\{v(a), v(b)\}$ and $v(0) = \infty$. We deviate from this convention since in the context of this paper, it is more natural to work with exponential valuations.

According to Proposition 3.4, the concept of Krull valuation can be generalized by considering seminorms of R in idempotent semirings S, which correspond to morphisms $R^{\mathrm{mon}} \to S^{\mathrm{pos}}$ of ordered blueprints. We will see in section 7.4 that the class of idempotent semirings plays a particular role for tropicalizations. This viewpoint can also be found in Macpherson's paper [38].

3.3. Characters. If the canonical inclusion $B^\bullet \to B^{\mathrm{mon}}$ is an isomorphism or if $S^{\mathrm{pos}} = \{0\}$, then a valuation $v : B \to S$ is nothing else than a monoid morphism $v^\bullet : B^\bullet \to S^\bullet$. Both of these hypothesis are satisfied for characters of an abelian group G in a field k, which is a group homomorphism $G \to k^\times$.

More precisely, if we define the monoid with zero $B = G \cup \{0\}$, then $B^\bullet \simeq B^{\mathrm{mon}}$. Since a field $S = k$ is with -1, we have $S^{\mathrm{pos}} = 0$. Since the image of 0 is determined, we see that the association

$$\begin{array}{ccc} \{\text{valuations } v : B \to k\} & \longrightarrow & \{\text{characters } \chi : G \to k\} \\ v : B \to F & \longmapsto & v^\bullet|_G : G \to k \end{array}$$

is a bijection. We can characterize unitary characters in $\mathbb{C}$ as valuations in the monoid $S = \mathbb{S}^1 \cup \{0\}$, i.e. the association

$$\begin{array}{ccc} \{\text{valuations } v : B \to S\} & \longrightarrow & \{\text{unitary characters } \chi : G \to \mathbb{C}\} \\ v : B \to F & \longmapsto & G \hookrightarrow B \xrightarrow{v^\bullet} S \hookrightarrow \mathbb{C} \end{array}$$

is a bijection.

4. Scheme theory

In this section, we introduce the geometric framework for scheme theoretic tropicalizations, as considered in this text. The central object of this theory is an ordered blue scheme, which is a generalization of a blue scheme, as introduced in [32]. Roughly speaking, an ordered blue scheme is a certain topological space together with a sheaf in OBlpr.

Since the focus of this paper lies in the application of this formalism to the tropicalization of classical varieties, we only include an exposition of this theory, which is, strictly speaking, not covered by the previous results in [32]. However, we would like to point out that the generalization from blueprints to ordered blueprints is a very harmless step, and that all proofs of [32] apply to the more general setting of ordered blueprints without major modifications.

Alternatively, one can extend the results of [32] to an ordered blueprint B in terms of the bijection $B^{\mathrm{core}} \to B$. We briefly comment on this in sections 4.3 and 4.5.

We advise the reader who encounters blue schemes for the first time to have a look at the examples in section 4 ("Basic definitions") of [33].

4.1. Localizations. Let B be an ordered blueprint with underlying monoid A and subaddition $\mathcal{R}$. A *multiplicative subset of* B is a multiplicatively closed subset S of A that contains 1. The *localization of* B *at* S is the ordered blueprint $S^{-1}B = S^{-1}A /\!\!/ \mathcal{R}_S$ where $S^{-1}A = \{\frac{a}{s} | a \in A, s \in S\}$ is the localization of the monoid A at S, i.e. $\frac{a}{s} = \frac{a'}{s'}$ if and only if there is a $t \in S$ such that $tsa' = ts'a$, and where

$$\mathcal{R}_S \ = \ \Big\langle \sum \frac{a_i}{1} \equiv \sum \frac{b_j}{1} \, \Big| \, \sum a_i \equiv \sum b_j \text{ in } B \Big\rangle.$$

The localization of B at S comes together with a canonical morphism $B \to S^{-1}B$ that sends a to $\frac{a}{1}$.

We say that a morphism $B \to C$ is a *localization* if there is a multiplicative subset S of B such that $C \simeq S^{-1}B$ and $B \to C$ corresponds to the canonical morphism $B \to S^{-1}B$ under this isomorphism. We say that $B \to C$ is a *finite localization* if the multiplicative subset S of B can be chosen to be finitely generated, i.e. one can choose finitely many generators $s_1, \dots, s_n$ in S such that every other element $t \in S$ is a product of powers of $s_1, \dots, s_n$. If S is generated by $s_1, \dots, s_n$, then $S^{-1}B = B[h^{-1}]$ where $h = s_1 \cdots s_n$ and $B[h^{-1}] = \tilde{S}^{-1}B$ for $\tilde{S} = \{h^i\}_{i \geqslant 0}$.

4.2. Ordered blueprinted spaces. An *ordered blueprinted space*, or for short an OBlpr-*space*, is a topological space X together with a sheaf $\mathcal{O}_X$ in OBlpr. In practice, we suppress the *structure sheaf* $\mathcal{O}_X$ from the notation and denote an OBlpr-space by the same symbol X as its *underlying topological space*. For every point x of X, the *stalk in* x is the colimit $\mathcal{O}_{X,x} = \operatorname{colim} \mathcal{O}_X(U)$ over the system of all open neighbourhoods U of x.

A *morphism of* OBlpr-*spaces* is a continuous map $\varphi : X \to Y$ between the underlying topological spaces together with a morphism $\varphi^{\#} : \varphi^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ of sheaves on X that is *local* in the following sense: for every $x \in X$ and $y = \varphi(y)$, the induced morphism $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ of stalks sends non-units to non-units. This defines the category OBlprSp of ordered blueprinted spaces.

Remark 4.1. Note that we will consider different types of spectra in sections 4.4, 6.1 and 6.2, which are based on different types of ideals. The unusual definition of a "local" morphism of OBlpr-spaces as given in this text has the advantage to apply to all these different types of spectra in the correct way.

4.3. Ideals. The following is a straight forward extension of the definitions in [32] from algebraic blueprints to ordered blueprints. Note that in contrast to [32], we use the term k-ideals in order to avoid conflicts with the notion of an ideal of a semiring in the literature, which differs from that of a k-ideal; also cf. section 6.2.

A *k-ideal* of an ordered blueprint B is a subset I such that $0 \in I$, such that $IB = I$ and such that $c + \sum a_i \equiv \sum b_j$ with $a_i, b_j \in I$ implies $c \in I$. A k-ideal I is *prime* if its complement $S = B - I$ is a multiplicative subset. The *localization of B at $\mathfrak{p}$* is $B_{\mathfrak{p}} = S^{-1}B$. A k-ideal I is *proper* if $I \subsetneq B$. A k-ideal I is *maximal* if it is proper and $I \subset J \subsetneq B$ implies $I = J$ for every other k-ideal J. An ordered blueprint B is *local* if it has a unique maximal ideal $\mathfrak{m}$ and if $B = B^{\times} \cup \mathfrak{m}$.

The localization $B_{\mathfrak{p}}$ at a prime ideal $\mathfrak{p}$ of B is a local blueprint with maximal ideal $\mathfrak{p}B_{\mathfrak{p}}$. Note that there are ordered blueprints B with a unique maximal k-ideal that is properly contained in the complement of $B^{\times}$; cf. Example 4.6. Since a proof of the following basic result is missing in the literature, we include it here.

Lemma 4.2. *Every maximal k-ideal of an ordered blueprint is prime.*

Proof. Let $\mathfrak{m}$ be a maximal k-ideal of an ordered blueprint B, and $a, b \in B$ with $ab \in \mathfrak{m}$, but $a \notin \mathfrak{m}$. We have to show that $b \in \mathfrak{m}$.

Since $\mathfrak{m}$ is maximal, the k-ideal generated by $\mathfrak{m} \cup \{a\}$ is B. We claim that $cb \in \mathfrak{m}$ for every $c \in B$. This holds clearly for elements c of $\mathfrak{m}$ and for $c = a$. If $db \in \mathfrak{m}$, then also $cb \in \mathfrak{m}$ for every multiple c of d. If we have a relation of the form $\sum d_i + c \equiv \sum e_j$ for which $d_i b, e_j b \in \mathfrak{m}$, then the relation $\sum d_i b + cb \equiv \sum e_j b$ implies that $cb \in \mathfrak{m}$. Hence the claim follows by induction. In particular, it holds for $c = 1$, i.e. we have $b = 1 \cdot b \in \mathfrak{m}$, as desired. $\square$

The following facts can be derived from the corresponding results for algebraic blueprints in [32] since a subset I of an ordered blueprint B is an k-ideal if and only if is an k-ideal of the algebraic core B^{core} of B. We define $B/I = B /\!\!/ \langle a \sim 0 | a \in I \rangle$ for any subset I of B. The surjection $\pi_I : B \to B/I$ is universal among all morphisms of ordered blueprints that map I to 0. A subset I of B is a k-ideal if and only if $I = \pi_I^{-1}(0)$. Consequently, $f^{-1}(0)$ is a k-ideal for every morphism

$B \to C$ of ordered blueprints. Note that a k-ideal $\mathfrak{p}$ is prime if and only if $B/\mathfrak{p}$ is *without zero divisors*, i.e. $ab = 0$ implies $a = 0$ or $b = 0$ for all $a,b \in B/\mathfrak{p}$. However, the quotient of an ordered blueprint B by a maximal ideal $\mathfrak{m}$ does not have to be a blue field; cf. Example 4.6.

4.4. The spectrum. Let B be an ordered blueprint. We define the *spectrum* $\operatorname{Spec} B$ *of* B as the following ordered blueprinted space. The topological space of $X = \operatorname{Spec} B$ consists of the prime k-ideals of B and comes with the topology generated by the *principal opens*

$$U_h \;=\; U_{B,h} \;=\; \{\mathfrak{p} \in \operatorname{Spec} B \,|\, h \notin \mathfrak{p}\}$$

where h varies through the elements of B. Note that the principal opens form a basis of the topology for X since $U_h \cap U_g = U_{gh}$.

Let U be an open subset of X. A *section on* U is a function $s : U \to \coprod_{\mathfrak{p}\in U} B_{\mathfrak{p}}$ such that $s(\mathfrak{p}) \in B_{\mathfrak{p}}$, such that there is a finite open covering $\{U_{h_i}\}$ of U by principal open subsets U_{h_i} and such that there are elements $a_i \in B[h_i^{-1}]$ whose respective images in $B_{\mathfrak{p}}$ equal $s(\mathfrak{p})$ whenever $\mathfrak{p} \in U_{h_i}$. The set of sections $O_X(U)$ on U comes naturally with the structure of an ordered blueprint. This defines the *structure sheaf* $\mathcal{O}_X$ of X and completes the definition of $X = \operatorname{Spec} B$ as an OBlpr-space.

As usual, a morphism $f : B \to C$ of ordered blueprints defines a morphism $f^* : \operatorname{Spec} C \to \operatorname{Spec} B$ of OBlpr-spaces by taking the inverse image of prime ideals and pulling back sections. This defines the contravariant functor

$$\operatorname{Spec}: \; \mathrm{OBlpr} \;\longrightarrow\; \mathrm{OBlprSp}.$$

We call OBlpr-spaces in the essential image of this functor *affine ordered blue schemes*.

4.5. Globalization. A complication in the theory of blue schemes is that the ordered blueprint of global sections $\Gamma B = \Gamma(X, \mathcal{O}_X)$ of $X = \operatorname{Spec} B$ is in general not isomorphic to B; cf. Example 4.6 or [32, Ex. 3.8]. In a more fancy way of speaking, not every ordered blueprint defines a sheaf on OBlpr with respect to the Grothendieck topology coming from topological coverings of the spectra of ordered blueprints. However, this complication can be resolved in terms of the restriction to a certain reflective subcategory of OBlpr, as we will explain in the following.

The bijection $B^{\mathrm{core}} \to B$ yields for every ordered blueprint B a homeomorphism $\operatorname{Spec} B \to \operatorname{Spec} B^{\mathrm{core}}$ by the definition of k-ideals and principal opens. This allows us to deduce the following statements from the corresponding results for algebraic blueprints; cf. [31] and [32] for details.

Note that the association $B \mapsto \Gamma B$ defines a functor $\mathrm{OBlpr} \to \mathrm{OBlpr}$. We denote by $\Gamma\mathrm{OBlpr}$ the essential image of this functor. The interpretation of elements of B as sections yields a morphism $\sigma_B : B \to \Gamma B$, which we call the *globalization of* B.

We say that an ordered blueprint B is *global* if it is contained in $\Gamma\mathrm{OBlpr}$. Examples of global ordered blueprints are local ordered blueprints, monoids and rings.

Theorem 4.3 ([32, Thm. 3.12])**.** *The globalization* $\sigma_B : B \to \Gamma B$ *of an ordered blueprint* B *induces an isomorphism* $\sigma_B^* : \operatorname{Spec}\Gamma B \to \operatorname{Spec} B$ *of ordered blueprinted spaces.*

This theorem has a number of implications, cf. section 7 in [31].

Corollary 4.4. *Let* B *be an ordered blueprint.*

(i) *The globalization* $\sigma_B : B \to \Gamma B$ *is an isomorphism if* B *is global.*
(ii) *Every morphism* $B \to C$ *into a global ordered blueprint* C *factors uniquely through the globalization* $\sigma_B : B \to \Gamma B$.
(iii) *The restriction* $\operatorname{Spec} : \Gamma\mathrm{OBlpr} \to \mathrm{OBlprSp}$ *is a fully faithful embedding.*
(iv) *The inclusion* $\iota : \Gamma\mathrm{OBlpr} \to \mathrm{OBlpr}$ *as a subcategory is right inverse and right adjoint to* $\Gamma : \mathrm{OBlpr} \to \Gamma\mathrm{OBlpr}$.

A consequence of these facts is the following.

Corollary 4.5. *A global ordered blueprint with unique maximal k-ideal is local.*

Proof. Let B be a global ordered blueprint with unique maximal k-ideal $\mathfrak{m}$. We have to show that $B = B^\times \cup \mathfrak{m}$. Since $\mathfrak{m}$ is the unique maximal k-ideal of B, every global section of B is an element of the stalk of $\operatorname{Spec} B$ at $\mathfrak{m}$, i.e. $\Gamma B = B_\mathfrak{m}$, which is a local ordered blueprint. Since B is global, $B = \Gamma B = B_\mathfrak{m}$ is local. □

Example 4.6. Consider the semiring $B = \mathbb{B}[T]/\!\!/\langle T+1 \equiv T \equiv T^2\rangle$, which fails to satisfy various properties that hold for rings. The underlying set of B is $\{0,1,T\}$ and its unit group is $B^\times = \{1\}$. Its proper ideals are $\{0\}$ and $\{0,T\}$, which are both prime, but only $\{0\}$ is a k-ideal; cf. section 6.2 for the definition of a (prime) ideal.

This shows the following effects: B has a unique maximal k-ideal $\mathfrak{m} = \{0\}$, but $B^\times \cup \mathfrak{m} = \{0,1\}$ does not equal B, i.e. B is not local.

The quotient $B/\mathfrak{m} \simeq B$ of B by its maximal k-ideal is not a blue field. Note that the quotient $B/\{0,T\}$ of B by its maximal ideal $\{0,T\}$ is the trivial semiring $\{0\}$.

The semiring B is not global since it contradicts the conclusion of Corollary 4.5. In more detail, the globalization $\sigma_B : B \to \Gamma B$ of B is the localization $B \to B_\mathfrak{m}$ at its maximal k-ideal $\mathfrak{m} = \{0\}$. Since $B_\mathfrak{m}$ contains a multiplicative inverse of T, the relation $T^2 \equiv T$ implies $T \equiv 1$ in $B_\mathfrak{m}$. This shows that $B_\mathfrak{m} \simeq \mathbb{B}$ and that $\sigma_B : B \to \Gamma B \simeq \mathbb{B}$ is the morphism that maps T to 1.

4.6. Ordered blue schemes. Every open subset U of an ordered blueprinted space X is naturally an ordered blueprinted space with respect to the restriction of the structure sheaf $\mathcal{O}_X$ to U. An *affine open of* X is an open subset that is isomorphic to the spectrum of an ordered blueprint. An *ordered blue scheme* is an ordered blueprinted space X such that every point has an affine open neighbourhood. A morphism of ordered blue schemes is a morphism of OBlpr-spaces. We denote the category of ordered blue schemes by $\mathrm{Sch}_{\mathbb{F}_1}$.

We collect some consequences of Theorem 4.3; cf. [32] for more details. The category $\mathrm{Sch}_{\mathbb{F}_1}$ contains all finite limits. The fibre product of ordered blue schemes is constructed as in usual scheme theory. In particular, the fibre product of a diagram $X \to Z \leftarrow Y$ of affine ordered blue schemes is represented by $\operatorname{Spec}\big(\Gamma X \otimes_{\Gamma Z} \Gamma Y\big)$. Note that $\operatorname{Spec} \mathbb{F}_1$ is a terminal object, which justifies the notation $\mathrm{Sch}_{\mathbb{F}_1}$.

More generally, we define for any ordered blueprint B the category Sch_B of *ordered blue B-schemes* as the category whose objects are morphisms $X \to \operatorname{Spec} B$ of ordered blue schemes and whose morphisms are morphisms $X \to Y$ of ordered blue schemes that commute with the structure maps $X \to \operatorname{Spec} B$ and $Y \to \operatorname{Spec} B$.

The global section functor $\Gamma : \mathrm{Sch}_{\mathbb{F}_1} \to \mathrm{OBlpr}$ is adjoint to $\operatorname{Spec} : \mathrm{OBlpr} \to \mathrm{Sch}_{\mathbb{F}_1}$, i.e. $\operatorname{Hom}(X, \operatorname{Spec} B) = \operatorname{Hom}(B, \Gamma X)$ for all ordered blueprints B and ordered blue schemes X, where we employ, by abuse of language, the same symbol for the global sections $\Gamma X = \Gamma(X, \mathcal{O}_X)$ as for the globalization ΓB of an ordered blueprint B.

Every morphism $\varphi : X \to Y$ of ordered blue schemes is *locally algebraic*, by which we mean that X and Y have affine open coverings $\{U_i\}$ and $\{V_i\}$, respectively, such that φ restricts for every i to a morphism $\varphi_i : U_i \to V_i$ between affine ordered blue schemes that is induced by a morphism $f_i : \Gamma V_i \to \Gamma U_i$ of ordered blueprints.

4.7. Stalks and residue fields. Let X be an ordered blue scheme and x a point of X. The *stalk of $\mathcal{O}_X$ in x* is the colimit $\mathcal{O}_{X,x} = \operatorname{colim} \mathcal{O}_X(U)$ over the system of all open neighbourhoods U of x. The stalk only depends on the subsets of an affine open neighbourhood of x, so we can assume that $X = \operatorname{Spec} B$ is affine and that $x = \mathfrak{p}$ is a prime ideal of B. In this case, we have $\mathcal{O}_{X,x} = B_\mathfrak{p}$. This means that the stalk $\mathcal{O}_{X,x}$ of x is a local ordered blueprint with a unique maximal ideal $\mathfrak{m}_x$.

The *residue field at* x is $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$, which is an ordered blue field. A morphism $\varphi : X \to Y$ of affine ordered blue schemes induces morphisms $\varphi_x : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ and $\kappa(y) \to \kappa(x)$ of ordered blueprints for every $x \in X$ and $y = \varphi(x)$.

4.8. Open and closed immersions. A morphism $\varphi : X \to Y$ of ordered blue schemes is an *open immersion* if it is an open topological embedding and if $\mathcal{O}_X$ is the restriction of $\mathcal{O}_Y$ to X. We say that $\varphi : X \to Y$ is isomorphic to another open immersion $\varphi : X' \to Y$ if there is an isomorphism $X \to X'$ commuting with φ and φ'. An *open (ordered blue) subscheme of* Y is an isomorphism class of open immersions into Y. Note that the open subschemes of Y correspond bijectively to the open subsets of Y.

A morphism $\varphi : X \to Y$ of ordered blue schemes is *affine* if for every open immersion $U \to Y$ from an affine ordered blue scheme U to Y, the inverse image $\varphi^{-1}(U) = U \times_Y X$ is affine. A morphism $\varphi : X \to Y$ of ordered blue schemes is a *closed immersion* if it is affine and if $\mathcal{O}_X(U) \to \mathcal{O}_Y(\varphi^{-1}(U))$ is surjective for every open immersion $U \to Y$ from an affine ordered blue scheme U to Y.

We say that $\varphi : X \to Y$ is isomorphic to another closed immersion $\varphi : X' \to Y$ if there is an isomorphism $X \to X'$ commuting with φ and φ'. A *closed (ordered blue) subscheme of* X is an isomorphism class of closed immersions into X.

Note that the image of a closed immersion $X \to Y$ does not have to be a closed subset of Y. For example, the diagonal embedding $\mathbb{A}^1_B \to \mathbb{A}^2_B$ is a closed immersion, but its image is not a closed subset.

4.9. Affine presentations. A useful tool for extending functors from ordered blueprints to ordered blue schemes are affine presentations. This are diagrams of affine ordered blue schemes and open immersions whose colimit is an ordered blue scheme. The existence of a colimit as an ordered blue scheme can be derived from the lack of monodromy of the diagram, which makes it possible to treat ordered blue scheme purely in terms of diagrams in the category $\mathrm{Aff}_{\mathbb{F}_1}$ of affine ordered blue schemes, which is anti-equivalent to ΓOBlpr. We review the definitions and some results from [31].

Let $\mathcal{A}$ be a category with finite products that is endowed with a Grothendieck pretopology. We say that a morphism in $\mathcal{A}$ is *open* if it occurs in a covering family. In the case $\mathcal{A} = \mathrm{Aff}_{\mathbb{F}_1}$ an open morphism is the same as an open immersion.

Let $\mathcal{U}$ be a diagram in $\mathcal{A}$. An object U of $\mathcal{U}$ is called a *maximal object* if any morphism in $\mathcal{U}$ with domain U is an isomorphism. We denote the set of maximal objects of $\mathcal{U}$ by $\mathcal{U}_{\max}$. We say that $\mathcal{U}$ is *with enough maximal objects* if for every object U of $\mathcal{U}$, there is a finite chain of morphisms $U \to U_1 \to \cdots \to U_n$ in $\mathcal{U}$ into a maximal object U_n.

Let U_0 and U_1 be two objects of $\mathcal{U}$. A *path in* $\mathcal{U}$ *from* U_0 *to* U_1 is a sequence

$$U_0 \xrightarrow{\quad\varphi_1\quad} V_1 \xrightarrow{\quad\varphi_2\quad} \cdots \xrightarrow{\quad\varphi_{n-1}\quad} V_{n-1} \xrightarrow{\quad\varphi_n\quad} U_1$$

of objects and morphisms in $\mathcal{U}$ whose arrows are allowed to have any orientation. Note that $\mathcal{V}$ is itself a diagram and comes with a map $\mathcal{V} \to \mathcal{U}$ of diagrams, which does not have to be injective. The limit $\lim \mathcal{V}$ of $\mathcal{V}$ in $\mathcal{A}$ comes with canonical projections $\lim \mathcal{V} \to U_0$ and $\lim \mathcal{V} \to U_1$, which we call the *beginning* and the *end of* $\mathcal{V}$, respectively.

A *monodromy-free diagram in* $\mathcal{A}$ is a diagram $\mathcal{U}$ in $\mathcal{A}$ such that for any object U in $\mathcal{U}$ and any path $\mathcal{V}$ in $\mathcal{U}$ from U to U, the canonical morphism

$$\lim\big(\lim \mathcal{V} \rightrightarrows U\big) \quad\longrightarrow\quad \lim \mathcal{V}$$

is an isomorphism where the two arrows $\lim \mathcal{V} \to U$ are the beginning and the end of $\mathcal{V}$. Note that a monodromy-free diagram is commutative.

An *affine presentation in* $\mathrm{Aff}_{\mathbb{F}_1}$ is a monodromy-free diagram $\mathcal{U}$ of open morphisms in $\mathcal{A}$ with enough maximal objects. A *morphism* $\mathcal{U} \to \mathcal{V}$ *of affine presentations* $\mathcal{U}$ *and* $\mathcal{V}$ is a family of morphisms $\{\varphi_U : U \to V(U)\}$ from all objects U in $\mathcal{U}$ to some objects $V(U)$ in $\mathcal{V}$ such there is a morphism $V(U_1) \to V(U_2)$ in $\mathcal{V}$ for every morphism $U_1 \to U_2$ in $\mathcal{U}$ such that the resulting square

$$\begin{tikzcd} U_1 \arrow[r] \arrow[d] & V(U_1) \arrow[d] \\ U_2 \arrow[r] & V(U_2) \end{tikzcd}$$

commutes.

Let $\mathcal{V}$ be an affine representation. A *refinement of* $\mathcal{V}$ is a morphism $\Phi : \mathcal{U} \to \mathcal{V}$ of affine presentations such that for all V in $\mathcal{V}$, the family $\{\Phi : U \to \Phi(U)\}$ with $\Phi(U) = V$ is a covering family of V.

The following theorem summarizes Theorem 7.4 and Proposition 7.5 from [31]. We extend the results from algebraic blue schemes to ordered blue schemes. Since the proofs apply literally to this more general setting, we omit them in this exposition.

Theorem 4.7. *In the following, all affine presentations are affine presentations in* $\mathrm{Aff}_{\mathbb{F}_1}$.

(i) *Every affine presentation* $\mathcal{U}$ *has a colimit* $X = \operatorname{colim} \mathcal{U}$ *in* $\mathrm{Sch}_{\mathbb{F}_1}$, *and every ordered blue scheme is the colimit of an affine presentation.*
(ii) *If* $X = \operatorname{colim} \mathcal{U}$, *then the canonical inclusion* $U \to X$ *is an open immersion for every* U *in* $\mathcal{U}$, *and* X *is covered by the maximal objects in* $\mathcal{U}$.
(iii) *Every morphism* $\Phi : \mathcal{U} \to \mathcal{V}$ *of affine presentations induces a morphism* $\varphi = \operatorname{colim} \Phi :$ $X \to Y$ *of ordered blue schemes between the respective colimits* $X = \operatorname{colim} \mathcal{U}$ *and* $Y =$ $\operatorname{colim} \mathcal{V}$, *and every morphism of ordered blue schemes is the colimit of a morphism of affine presentations.*
(iv) *The colimit of a refinement* $\Phi : \mathcal{U} \to \mathcal{V}$ *is an isomorphism of ordered blue schemes. Given a morphism* $\varphi : \operatorname{colim} \mathcal{U} \to \operatorname{colim} \mathcal{V}$ *of ordered blue schemes where* $\mathcal{U}$ *and* $\mathcal{V}$ *are affine presentations, there exists a refinement* $\Psi : \mathcal{U}' \to \mathcal{U}$ *and a morphism* $\Phi : \mathcal{U}' \to \mathcal{V}$ *of affine presentations such that* $\varphi \circ \operatorname{colim} \Psi = \operatorname{colim} \Phi$. *If* φ *is an isomorphism, then* Φ *is a refinement.*

We say that an endofunctor $\mathcal{F} : \mathrm{OBlpr} \to \mathrm{OBlpr}$ *preserves finite localizations* if $\mathcal{F}(B) \to \mathcal{F}(C)$ is a finite localization for every finite localization $B \to C$. We say that $\mathcal{F}$ *preserves covering families* if for every ordered blueprint B and every covering of $X = \operatorname{Spec} B$ by affine open subschemes $U_i = \operatorname{Spec} B_i$, the induced morphisms $\operatorname{Spec} \mathcal{F}(B_i) \to \operatorname{Spec} \mathcal{F}(B)$ are open immersions and $\operatorname{Spec} \mathcal{F}(B)$ is covered by the open subschemes $\operatorname{Spec} \mathcal{F}(B_i)$.

The following fact is Lemma 1.3 in [31].

Lemma 4.8. *Let* $\mathcal{G} : \mathrm{Aff}_{\mathbb{F}_1} \to \mathrm{Aff}_{\mathbb{F}_1}$ *be a functor that commutes with fibre products and that preserves covering families. Then there exists a unique functor* $\overline{\mathcal{G}} : \mathrm{Sch}_{\mathbb{F}_1} \to \mathrm{Sch}_{\mathbb{F}_1}$ *such that for all morphisms* $\Phi : \mathcal{U} \to \mathcal{V}$ *of affine presentations in* $\mathrm{Aff}_{\mathbb{F}_1}$, *we have a natural identification* $\overline{\mathcal{G}}(\operatorname{colim} \Phi) = \operatorname{colim} \mathcal{G}(\Phi)$. *In particular, this yields* $\overline{\mathcal{G}}(\operatorname{colim} \mathcal{U}) = \operatorname{colim} \mathcal{G}(\mathcal{U})$.

4.10. Endofunctors. Some of the subcategories $\mathcal{C}$ of OBlpr from section 2.10 extend to subcategories of $\mathrm{Sch}_{\mathbb{F}_1}$ as reflective subcategories.

Lemma 4.9. *The endofunctors* $(-)^{+}$, $(-)^{\mathrm{hull}}$, $(-)^{\mathrm{inv}}$, $(-)^{\mathrm{idem}}$, $(-)^{\mathrm{pos}}$, $(-)^{\mathrm{conic}}$, $(-)^{\mathrm{core}}$, $(-)^{\bullet}$, $(-)^{\mathrm{mon}}$ *and* $(-)^{\mathrm{padd}}$ *preserve finite localizations. All of these endofunctors but the last three preserve covering families.*

Proof. Let S be a finitely generated multiplicative subset of B. If $\mathcal{F}$ if one of the functors $(-)^+$, $(-)^{\mathrm{hull}}$, $(-)^{\mathrm{inv}}$, $(-)^{\mathrm{idem}}$, $(-)^{\mathrm{pos}}$ or $(-)^{\mathrm{conic}}$, then $\mathcal{F}(B)$ comes together with a canonical morphism $\varphi : B \to \mathcal{F}(B)$, and $\mathcal{F}(S^{-1}B) = \big(\varphi(S)\big)^{-1}\mathcal{F}(B)$.

If $\mathcal{F}$ is $(-)^{\mathrm{core}}$, $(-)^{\bullet}$, $(-)^{\mathrm{mon}}$ or $(-)^{\mathrm{padd}}$, then $\mathcal{F}(B)$ comes together with a bijection $\mathcal{F}(B) \to B$, and $\mathcal{F}(S^{-1}B) = S^{-1}\mathcal{F}(B)$.

Note that it suffices to verify the preservation of covering families for families of principal opens. Since $\mathcal{F}$ preserves finite localizations, it sends principal opens $U_h = \operatorname{Spec} B[h^{-1}]$ of the spectrum $\operatorname{Spec} B$ of an ordered blueprint B to principal opens $U_{\tilde{h}}$ of $\operatorname{Spec}\mathcal{F}(B)$ where $\tilde{h}$ can be identified with (the image of) h in $\mathcal{F}(B)$ for each functor $\mathcal{F}$ under consideration.

Let $\{U_{h_i}\}$ be a family of principal open subsets of $\operatorname{Spec} B$. By definition, U_{h_i} consists of all prime ideals $\mathfrak{p}$ that do not contain h_i. Thus $\mathfrak{p} \notin \bigcup U_{h_i}$ if and only if $\langle h_i\rangle \subset \mathfrak{p}$. Since every prime ideal is contained in a maximal ideal, it follows that $\operatorname{Spec} B = \bigcup U_{h_i}$ if and only if $\langle h_i\rangle = \langle 1\rangle$, i.e. there exists a relation of the form $1+\sum b_k h_{i_k} \equiv \sum c_l h_{i_l}$ with $b_k, c_l \in B$. It is easily verified for each of the functors $(-)^+$, $(-)^{\mathrm{hull}}$, $(-)^{\mathrm{inv}}$, $(-)^{\mathrm{idem}}$, $(-)^{\mathrm{pos}}$, $(-)^{\mathrm{conic}}$ and $(-)^{\mathrm{core}}$ that it preserves this type of relation. This finishes the proof of the lemma. □

Note that the endofunctors $(-)^{\bullet}$, $(-)^{\mathrm{mon}}$ and $(-)^{\mathrm{padd}}$ fail to preserve covering families since they do not preserve relations of the form $1+\sum b_k h_{i_k} \equiv \sum c_l h_{i_l}$.

Given an endofunctor $\mathcal{F}$ on OBlpr, we might attempt to extend it to an endofunctor on $\mathrm{Sch}_{\mathbb{F}_1}$ by covering an ordered blue scheme X by affine opens U_i, applying $\mathcal{F}$ to the ΓU_i and glueing together $\operatorname{Spec}\mathcal{F}(\Gamma U_i)$ to get $\mathcal{F}(X)$. Though this process is in general not independent from the chosen covering, this is the case for the functors considered in the following theorem.

Theorem 4.10. *The functors $(-)^+$, $(-)^{\mathrm{hull}}$, $(-)^{\mathrm{inv}}$, $(-)^{\mathrm{idem}}$, $(-)^{\mathrm{pos}}$, $(-)^{\mathrm{conic}}$ and $(-)^{\mathrm{core}}$ extend to endofunctors of the category of ordered blue schemes and satisfy the following properties.*

(i) *Let $\dagger$ be one of $+$, hull, inv, idem, pos or conic. Then $(-)^\dagger : \mathrm{Sch}_{\mathbb{F}_1} \to \mathrm{Sch}_{\mathbb{F}_1}$ is idempotent and comes with a canonical morphism $X^\dagger \to X$ for every ordered blue scheme X. Let $\mathrm{Sch}_{\mathbb{F}_1}^\dagger$ be the essential image of $(-)^\dagger$ and $\iota : \mathrm{Sch}_{\mathbb{F}_1}^\dagger \to \mathrm{Sch}_{\mathbb{F}_1}$ the inclusion functor. Then the restriction $(-)^\dagger : \mathrm{Sch}_{\mathbb{F}_1} \to \mathrm{Sch}_{\mathbb{F}_1}^\dagger$ is right adjoint and left inverse to ι.*

(ii) *The functor $(-)^{\mathrm{core}} : \mathrm{Sch}_{\mathbb{F}_1} \to \mathrm{Sch}_{\mathbb{F}_1}$ is idempotent and comes with a canonical morphism $X \to X^{\mathrm{core}}$ for every ordered blue scheme X. Let $\mathrm{Sch}_{\mathbb{F}_1}^{\mathrm{core}}$ be the essential image of $(-)^{\mathrm{core}}$ and $\iota : \mathrm{Sch}_{\mathbb{F}_1}^{\mathrm{core}} \to \mathrm{Sch}_{\mathbb{F}_1}$ the inclusion functor. Then the restriction $(-)^{\mathrm{core}} : \mathrm{Sch}_{\mathbb{F}_1} \to \mathrm{Sch}_{\mathbb{F}_1}^{\mathrm{core}}$ is left adjoint and left inverse to ι.*

Proof. We begin with verifying that all functors in question commute with fibre products. For $\dagger$ in $\{+,\mathrm{hull},\mathrm{inv},\mathrm{idem},\mathrm{pos},\mathrm{conic}\}$, we have for global ordered blueprints B and C with $C = C^\dagger$ that $\mathrm{Hom}(B,C) = \mathrm{Hom}(\Gamma(B^\dagger),C)$. This means $(-)^\dagger : \mathrm{OBlpr} \to \mathrm{OBlpr}^\dagger$ is left adjoint to the embedding $\mathrm{OBlpr}^\dagger \to \mathrm{OBlpr}$, and that the corresponding functor $(-)^\dagger : \mathrm{Aff}_{\mathbb{F}_1} \to \mathrm{Aff}_{\mathbb{F}_1}^\dagger$ between the dual categories is a right adjoint and therefore commutes with fibre products. It is obvious that $(-)^{\mathrm{core}} : \mathrm{OBlpr} \to \mathrm{OBlpr}$ commutes with tensor products, and consequently, its geometric version $(-)^{\mathrm{core}} : \mathrm{Aff}_{\mathbb{F}_1} \to \mathrm{Aff}_{\mathbb{F}_1}$ commutes with fibre products.

Since all functors $(-)^\dagger$ in question preserve covering families by Lemma 4.9 and commute with fibre products, Lemma 4.8 extends $(-)^\dagger$ to a functor $(-)^\dagger : \mathrm{Sch}_{\mathbb{F}_1} \to \mathrm{Sch}_{\mathbb{F}_1}$.

The adjointness properties follow from the corresponding properties of $(-)^\dagger$ on affine ordered blue schemes and the fact that every morphism between ordered blue schemes comes from a morphism of affine presentations; cf. Theorem 4.7. □

As a consequence, the essential images of the functors considered in Theorem 4.10 define a variety of subcategories of $\mathrm{Sch}_{\mathbb{F}_1}$. If $\dagger$ is in $\{+,\mathrm{hull},\mathrm{inv},\mathrm{idem},\mathrm{pos},\mathrm{conic},\mathrm{core}\}$, then $\mathrm{Sch}_{\mathbb{F}_1}^\dagger$ is (co)reflective in $\mathrm{Sch}_{\mathbb{F}_1}$, by (i) and (ii). The essential image of the global section functor

$\Gamma : \mathrm{Sch}_{\mathbb{F}_1}^{\dagger} \to \Gamma\mathrm{OBlpr}$ is the (co)reflective subcategory $\Gamma\mathrm{OBlpr}^{\dagger}$ of $\Gamma\mathrm{OBlpr}$ that consists of all global ordered blueprints B with $B^{\dagger} \simeq B$. In particular, note that for any ordered blueprint B with $B^{\dagger} \simeq B$, its globalization ΓB is in $\Gamma\mathrm{OBlpr}^{\dagger}$.

If $\dagger$ is in $\{\bullet, \mathrm{mon}, \mathrm{padd}\}$, then we define $\mathrm{Sch}_{\mathbb{F}_1}^{\dagger}$ as the full subcategory of $\mathrm{Sch}_{\mathbb{F}_1}$ of ordered blue schemes that can be covered by spectra of ordered blueprints in $\mathrm{OBlpr}^{\dagger}$. Note that in cases considered here, the right adjoint $(-)^{\dagger} : \mathrm{OBlpr} \to \mathrm{OBlpr}^{\dagger}$ to the embedding $\mathrm{OBlpr}^{\dagger} \to \mathrm{OBlpr}$ as subcategory does not extend to a left adjoint to the embedding $\mathrm{Sch}_{\mathbb{F}_1}^{\dagger} \to \mathrm{Sch}_{\mathbb{F}_1}$.

An *ordered blue B-scheme* is an ordered blue scheme X together with a morphism $X \to \mathrm{Spec}\, B$, called the *structure morphism*. A B-morphism between ordered blue B-schemes X and Y is a morphism $X \to Y$ that commutes with the structure morphisms of X and Y. We denote the category of ordered blue B-schemes by Sch_B. Note that a morphism $B \to C$ of ordered blueprints induces the functor $- \otimes_B C : \mathrm{Sch}_B \to \mathrm{Sch}_C$ that maps an ordered blue B-scheme X to the ordered blue C-scheme $X_C = X \times_{\mathrm{Spec}\, B} \mathrm{Spec}\, C$.

We list some important subcategories of $\mathrm{Sch}_{\mathbb{F}_1}$ in the following:

- the subcategory $\mathrm{Sch}_{\mathbb{F}_1}^{\mathrm{alg}}$ of blue schemes;
- the subcategory $\mathrm{Sch}_{\mathbb{F}_1}^{\bullet}$ of monoid schemes;
- the subcategory $\mathrm{Sch}_{\mathbb{N}}^{+}$ of semiring schemes;
- the subcategory $\mathrm{Sch}_{\mathbb{Z}}^{+}$ of usual schemes.

5. Rational points

In this section, we endow the set of T-rational points $X(T) = \mathrm{Hom}_k(\mathrm{Spec}\, T, X)$ with a topology for every ordered blue k-scheme X, coming from a topology for the ordered blue k-algebra T.

5.1. The affine topology. Let k be an ordered blueprint and T an *ordered blue k-algebra*, which is an ordered blueprint together with a morphism $k \to T$. We assume that T is a global ordered blueprint and that it is equipped with a topology, which, a priori, we do not assume to satisfy any compatibility with the structure of T as an ordered blueprint.

Let B be an ordered blue k-algebra and $h_B(T) = \mathrm{Hom}_k(B, T)$ the set of k-linear morphisms. The *affine topology for $h_B(T)$* is the compact-open topology where we consider B as a discrete ordered blueprint. Since the compact subsets of B are precisely the finite subsets, the compact-open topology on $h_B(T)$ is generated by open subsets of the form $U_{a,V} = \{ f : B \to T \,|\, f(a) \in V \}$ where $a \in B$ and $V \subset T$ is an open subset. In other words, the affine topology on $h_B(T)$ is the coarsest topology such that the maps

$$\begin{array}{rccc} \mathrm{ev}_a : & \mathrm{Hom}(B,T) & \longrightarrow & T \\ & (f : B \to T) & \longmapsto & f(a) \end{array}$$

are continuous for all $a \in B$.

5.2. The fine topology. For an arbitrary ordered blue scheme X, we endow $X(T)$ with the *fine topology*, which is the finest topology such that for all morphisms $\alpha : U \to X$ from an affine ordered blue k-scheme U to X, the induced map $\alpha_T : U(T) \to X(T)$ is continuous with respect to the affine topology on $U(T) = h_{\Gamma U}(T)$. Note that $U(T) = h_{\Gamma U}(T)$ since T is global.

The definition of the fine topology is justified in [35]: the fine topology coincides with the known topologies in the cases that X is a k-scheme and $T = k$ is a topological field or T is the adele ring of a global field k.

5.3. Functoriality. The following properties can be shown by similar arguments as used to prove the analogous statements for rings and schemes in [35]. Since the adaptation of these arguments requires some slight modifications, we provide a proof.

Lemma 5.1. *Let k be an ordered blueprint and T a global ordered blue k-algebra with topology.*

(i) *Let B be an ordered blue k-algebra. Then the affine topology for $h_B(T)$ is functorial in both B and T.*

(ii) *Let $X = \operatorname{Spec} B$ be an affine ordered blue k-scheme where $B = \Gamma X$ is global. Then the fine topology and the affine topology for $X(B) = h_B(T)$ coincide.*

(iii) *Let X be an ordered blue k-schemes. Then the fine topology for $X(T)$ is functorial in both X and T.*

Proof. We begin with (i). Since the topology of $h_B(T)$ is generated by open subsets of the form

$$U_{V,a} \quad = \quad \{ f : A \to R \,|\, f(a) \in V \}$$

with $a \in B$ and $V \subset T$ open, it suffices to verify that the inverse images of such open subsets are open to verify the continuity of the maps in question.

Let $g : C \to B$ a homomorphism of ordered blue k-algebras and $g^* : h_B(T) \to h_C(T)$ the pullback of morphisms. It is immediate that $(g^*)^{-1}(U_{V,b}) = U_{V,g(b)}$, which shows that g^* is continuous.

Let $f : T \to S$ be a continuous homomorphism of ordered blue k-algebras with topologies and $f_* : h_B(T) \to h_B(S)$ that pushforward of morphisms. It is immediate that $f_*^{-1}(U_{V,a}) = U_{f^{-1}(V),a}$, which shows that f_* is continuous. This verifies (i).

We continue with (ii). The identity morphism $\mathrm{id} : X \to X$ yields a continuous map $\mathrm{id}_T : X(T) \to X(T)$ with respect to the affine topology for the domain and the fine topology for the image. This shows that the affine topology is finer than the fine topology.

Conversely, note that every k-linear morphism $\alpha : U \to X$ factors through the identity $\mathrm{id} : X \to X$. We have already proven that the map $U(T) \to X(T)$ is continuous with respect to the affine topology for both domain and image. This shows that the fine topology is at least as fine as the affine topology. We conclude that both topologies coincide. This is the claim of (iii).

We continue with (iii). Let $\varphi : X \to Y$ be a morphism of ordered blue k-schemes and $\varphi_T : X(T) \to Y(T)$ the induced map. Let $W \subset Y(T)$ be open. We have to show that $Z = \varphi_T^{-1}(W)$ is open in $X(T)$, which is the case if $\alpha_T^{-1}(Z)$ is open in $U(T)$ for every morphism $\alpha : U \to X$ from an affine ordered blue k-scheme U to X.

Since $\varphi \circ \alpha : U \to Y$ is a k-morphism from the affine ordered blue scheme U to Y, the inverse image $\alpha_T^{-1}(Z) = (\varphi \circ \alpha)_T^{-1}(W)$ of W in $U(T)$ is indeed open. This shows that the fine topology of $X(T)$ is functorial in X.

Let $f : T \to S$ be a continuous homomorphism of ordered blue k-algebras with topologies and $f_X : X(T) \to X(S)$ the induced map. Let $W \subset X(S)$ be open. We have to show that $Z = f_X^{-1}(W)$ is open in $X(T)$, which is the case if $\alpha_T^{-1}(Z)$ is open in $U(T)$ for every morphism $\alpha : U \to X$ from an affine k-scheme U to X.

Since the affine topology is functorial, the homomorphism $f : T \to S$ induces a continuous map $f_U : U(T) \to U(S)$. Since $\alpha_S^{-1}(W)$ is open in $U(S)$, the inverse image $\alpha_T^{-1}(Z) = f_U^{-1}(\alpha_S^{-1}(W))$ is open in $U(T)$. This shows that the fine topology of $X(T)$ is functorial in T. $\square$

5.4. Properties of the fine topology. For suitable T, it is possible to describe the fine topology on $X(T)$ in terms of coverings by affine open subschemes. For formulating such a result, we require the following definitions. A *topological ordered blueprint* is an ordered blueprint T with a topology such that the multiplication map $T \times T \to T$ is continuous. A *topological semiring* is a semiring T with a topology such that multiplication and addition define continuous maps $T \times T \to T$. A topological ordered blueprint T is *with open unit group* if the multiplicative group $T^\times$ of invertible elements in T forms an open subset of T and if the multiplicative inversion $T^\times \to T^\times$ is a continuous map.

Theorem 5.2. *Let k be an ordered blueprint and T a local topological ordered blue k-algebra with open unit group. Then T satisfies the following properties.*

- *(F1) For every ordered blue k-algebra B and $X = \operatorname{Spec} B$, the canonical bijection $X(T) \to h_B(T)$ is a homeomorphism.*
- *(F2) The functor $X \mapsto X(T)$ commutes with limits.*
- *(F3) The canonical bijection $\mathbb{A}^1_k(T) \to T$ is a homeomorphism.*
- *(F4) An open immersion $Y \to X$ of ordered blue k-schemes yields an open embedding $Y(T) \to X(T)$ of topological spaces.*
- *(F5) A covering $X = \bigcup U_i$ of an ordered blue k-scheme X by open subschemes yields a covering $X(T) = \bigcup U_i(T)$ by open subspaces.*
- *(F6) A closed immersion $Y \to X$ of ordered blue k-schemes yields an embedding $Y(T) \to X(T)$ of topological spaces.*

If in addition, T is a topological semiring, then T satisfies the following property.

- *(F7) The canonical bijection $X^+(T) \to X(T)$ is a homeomorphism for every ordered blue k-scheme X.*

If in addition, T is Hausdorff, then T satisfies the following stronger version of (F6).

- *(F8) A closed immersion $Y \to X$ of ordered blue k-schemes yields a closed embedding $Y(T) \to X(T)$ of topological spaces.*

Before we turn to the proof of the theorem, we point out that the theorem applies to the main example of interest for tropicalizations and analytifications.

Example 5.3 (Tropical numbers)**.** The tropical numbers $\mathbb{T}$ inherit the real topology from the identification $\mathbb{T} = \mathbb{R}_{\geqslant 0}$. With this topology, $\mathbb{T}$ is a local topological Hausdorff semiring with open unit group. Thus $\mathbb{T}$ satisfies (F1)–(F8).

Proof. The strategy of the proof is as follows. As a first step, we will establish certain analogues (A2)–(A6) of (F2)–(F6) for the affine topology. We will employ these properties of the affine topology to prove (F1), which implies (F2)–(F6) for affine ordered blue k-schemes. With this, we will be able to prove (F2)–(F6) in full generality. Finally, we will verify (F7) and (F8).

We begin with the proof of the following analogue of (F6) for the affine topology.

- (A6) A surjection $B \to C$ of ordered blue k-algebras yields an embedding $h_C(T) \to h_B(T)$ of topological spaces.

A basic open of $h_C(T) = \operatorname{Hom}_k(C,T)$ is of the form $U_{b,V} = \{g : C \to T | g(b) \in V\}$ with $b \in C$ and $V \subset T$ open. Since $B \to C$ is surjective, $b = f(a)$ for some $a \in B$. Note that $h_C(T) \to h_B(T)$ is an inclusion, thus

$$U_{b,V} \ = \ U_{f(a),V} \ = \ \{g : B \to T \,|\, g(a) \in V \text{ and } g \text{ factors through } C\} \ = \ U_{a,V} \ \cap \ h_C(T).$$

This shows that $h_C(T) \to h_B(T)$ is a topological embedding and concludes the proof of (A6).

We continue with the proof of the following analogue of (F2) for the affine topology.

- (A2) The functor $B \mapsto h_B(T)$ commutes with colimits.

Let $\mathcal{D}$ be a diagram of ordered blue k-algebras with colimit B. We have to show that the canonical bijection $\Psi : h_B(T) \to h_{\operatorname{colim} \mathcal{D}}(T)$ is a homeomorphism. Note that the Yoneda embedding commutes with colimits, i.e. $h_{\operatorname{colim} \mathcal{D}} = \lim h_{\mathcal{D}}$ where $h_{\mathcal{D}}$ is the diagram of functors on $\mathrm{Alg}_k^{\mathrm{ob}}$ defined by $\mathcal{D}$.

The continuity of Ψ is easily verified: the canonical projections $\pi_i : h_B \to h_{C_i}$ to objects C_i of $\mathcal{D}$ induce continuous maps $\pi_{i,T} : h_B(T) \to h_{C_i}(T)$ that commute with all maps $\varphi_T : h_{C_i}(T) \to h_{C_j}(T)$ coming from morphisms $\varphi : C_j \to C_i$ in $\mathcal{D}$. These maps induce the canonical bijection $\Psi : h_B(T) \to \lim h_{\mathcal{D}}(T)$, which, consequently, is a continuous map.

We proceed to show that Ψ is open. Since every colimit can be expressed in terms of coequalizers and coproducts, it is enough to proof the openness of Ψ for these particular types of limits.

The coequalizer $E = \operatorname{coeq}(f,g)$ of two morphisms $f,g : C \to D$ comes with a surjection $D \to C$. By (A6), $h_E(T) \to h_D(T)$ is a topological embedding. Since the topological equalizer $\operatorname{eq}(f_T, g_T)$ of $f_T, g_T : h_D(T) \to h_C(T)$ has the subspace topology of $h_D(T)$, the canonical bijection $h_E(T) \to \operatorname{eq}(f_T, g_T)$ is a homeomorphism.

The coproduct of a family $(B_i)_{i\in I}$ of ordered blue k-algebras is the (possibly infinite) tensor product $B = \bigotimes B_i$ over k, whose elements are tensors $a = a_{i_1} \otimes \cdots \otimes a_{i_n}$ with coordinates in a finite set of factors $B_{i_1}, \dots, B_{i_n}$. Thus a basic open of $h_B(T)$ is of the form $U_{a,V}$ where $a \in \bigotimes B_i$ and $V \subset T$ is an open. We have

$$\Psi(U_{a,V}) \ = \ \{ (f_i : B_i \to T) \,|\, \prod_{k=1}^{n} f_{i_k}(a_{i_k}) \in V \} \ = \ \{ (f_i : B_i \to T) \,|\, \big(f_{i_1}(a_{i_1}, \dots, f_{i_n}(a_{i_n}) \big) \in \mu_n^{-1}(V) \}$$

where $\mu_n : T^n \to T$ is the n-fold multiplication. Since the multiplication of T is continuous, $\mu_n^{-1}(V)$ is open in T^n and can be covered by basic open subsets $V_{k,1} \times \cdots \times V_{k,n}$ where k varies in some index set I. Thus we have

$$\Psi(U_{a,V}) \ = \ \bigcup_{k\in I} \{ (f_i : B_i \to T) \,|\, f_{i_l}(a_{i_l}) \in V_{k,l} \text{ for } l = 1, \dots, n \} \ = \ \bigcup_{k\in I} \bigcap_{l=1}^{n} \pi_{i_l}^{-1}\big(U_{f_{i_l}(a_{i_l}), V_{k,l}}\big)$$

where $\pi_{i_l} : \prod X_i(T) \to X_{i_l}$ is the canonical projection. This shows that $\Psi(U_{a,V})$ is open in $\prod X_i(T)$, as desired. This finishes the proof of (A2).

We continue with the proof of the following analogue of (F3) for the affine topology.

(A3) The canonical bijection $h_{k[t]}(T) \to T$ is a homeomorphism.

The canonical bijection $\Phi : h_{k[t]}(T) \to T$ sends a homomorphism $f : k[t] \to T$ to $f(t)$. Given an open subset $V \subset T$, we have $\Phi^{-1}(V) = \{ f \in \mathbb{A}^1_k(T) | f(t) \in V \}$, which is the basic open subset $U_{t,V}$ of $\mathbb{A}^1_k(T)$. Thus Φ is continuous.

A basic open of $h_{k[t]}(T) = \operatorname{Hom}_k(k[t], T)$ is of the form $U_{a,V}$ with $a \in k[t]$ and $V \subset T$ open. Every element a of $k[t]$ is of the form ct^i for some $c \in k$ and some $i \geqslant 0$. Since the multiplication of T is continuous, the evaluation of the monomial $a = ct^i$ in elements of T defines a continuous map $a : T \to T$. The inverse image $W = a^{-1}(V)$ is open in T, and thus

$$U_{a,V} \ = \ \{ f : k[t] \to T \,|\, f(t) \in a^{-1}(V) \} \ = \ U_{t,W}$$

is mapped to the open subset $\Phi(U_{a,V}) = W$ of T. This completes the proof of (A3).

We continue with the proof of the following analogue of (F4) for the affine topology.

(A4) The localization $B \to B[h^{-1}]$ of an ordered blue k-algebra B at an element $h \in B$ yields an open embedding $h_{B[h^{-1}]}(T) \to h_B(T)$ of topological spaces.

As a first case, we consider the localization $k[t] \to k[t^{\pm 1}]$ and the inclusion $\iota_T : h_{k[t^{\pm1}]}(T) \to h_{k[t]}(T)$. As sets, $h_{k[t^{\pm1}]}(T) = T^\times$, and by (A3), the canonical bijection $h_{k[t]}(T) \to T$ is a homeomorphism. Since T is with open unit group, $T^\times$ is an open subset of T and the image

$$\iota_T(U_{at^i,V}) \ = \ \{ f : k[t] \to T \,|\, f(at^i) \in V, f(t) \in T^\times \} \ = \ U_{at^i,V} \cap U_{t,T^\times}$$

of a basic open $U_{at^i,V} \subset h_{k[t^{\pm1}]}(T)$ is open in $h_{k[t]}(T)$ for $at^i \in k[t]$ and $V \subset T$ open. In particular, we can assume that $V \subset T^\times$.

If $at^{-i} \in k[t^{\pm1}]$ is in the complement of $k[t]$, i.e. if $-i < 0$, then we first observe that $U_{at^{-i},V} = U_{t^{-i},a^{-1}(V)}$ where $a^{-1}(V) = \{ b \in T | ab \in V \}$ is open in T since the multiplication of T is continuous. Thus we might assume that $a = 1$. Since the inversion $i : T^\times \to T^\times$, sending b to b^{-1} is

continuous, $i^{-1}(V)$ is open in T. Thus the image of $U_{t^{-i},V} = U_{t^i,i^{-1}(V)}$ is the basic open $U_{t^i,i^{-1}(V)}$ of $h_{k[t]}(T)$. This shows that ι_T is an open topological embedding.

In the general case of a localization $B \to B[h^{-1}]$, we have $B[h^{-1}] = B \otimes_{k[t]} k[t^{\pm 1}]$ with respect to the k-linear morphism $k[t] \to B$ that maps t to h. By (A2), $h_{B[h^{-1}]}(T)$ is homeomorphic to $h_B(T) \times_{h_{k[t]}(T)} h_{k[t^{\pm 1}]}(T)$, i.e. $\iota'_T : h_{B[h^{-1}]}(T) \to h_B(T)$ is the base change of the open topological embedding $\iota_T : h_{k[t^{\pm 1}]}(T) \to h_{k[t]}(T)$ along $h_B(T) \to h_{k[t]}(T)$, and therefore ι'_T itself is an open topological embedding, as desired. This concludes the proof of (A4).

We continue with the proof of the following analogue of (F5) for the affine topology.

(A5) A family of localizations $\{B \to B[h_i^{-1}]\}$ such that $\operatorname{Spec} B$ is covered by $\operatorname{Spec} B[h_i^{-1}]$ yields a covering $h_B(T) = \bigcup h_{B[h_i^{-1}]}(T)$ by open subspaces.

Since T is local, every morphism $\operatorname{Spec} T \to \operatorname{Spec} B$ factors through one of the principal open subschemes $\operatorname{Spec} B[h_i^{-1}]$. Thus we have $h_B(T) = \bigcup h_{B[h_i^{-1}]}(T)$ as sets. By (A4), the subsets $h_{B[h_i^{-1}]}(T)$ of $h_B(T)$ are indeed open subspaces. This concludes the proof of (A5).

We turn to the proof of (F1). By Lemma 5.1, we know that the canonical bijection $X(T) \to h_{\Gamma B}(T)$ is a homeomorphism. Thus we have to show that the bijection $\Theta : h_{\Gamma B}(T) \to h_B(T)$ that is induced by $\sigma : B \to \Gamma B$ is a homeomorphism.

Its continuity is easily verified: let $U_{a,V}$ be a basic open of $h_B(T)$ where $a \in B$ and $V \subset T$ is open. Then $\Theta^{-1}(U_{a,V})$ is the basic open $U_{\sigma(a),V}$ of $h_{\Gamma B}(T)$. Thus Θ is continuous.

We are left with showing that Θ is open. Consider a basic open $U_{a,V}$ of $h_{\Gamma B}(T)$ for $a \in \Gamma B$ and $V \subset T$ open. The global section a on $\operatorname{Spec} B$ can be represented by a tuple of elements $a_i/h_i \in B[h_i^{-1}]$ where the collection of principal open subschemes $\operatorname{Spec} B[h_i^{-1}]$ cover $\operatorname{Spec} B$ and such that $a_i/h_i = a_j/h_j$ in $B[(h_ih_j)^{-1}]$.

By (A5), $h_B(T) = \bigcup h_{B[h_i^{-1}]}(T)$ is a covering of $h_B(T)$ by open subspaces. Thus $\Theta(U_{a,V})$ is open in $h_B(T)$ if and only if the intersection $\Theta(U_{a,V}) \cap h_{B[h_i^{-1}]}(T)$ is open in $h_{B[h_i^{-1}]}(T)$ for all i. This intersection consists of all functions $f : B \to T$ such that the extension $\Gamma f : \Gamma B \to T$ maps a to V and such that $f(h_i) \in T^\times$. The latter condition means that f factors through $B[h_i^{-1}]$, which lets us identify $\Theta(U_{a,V}) \cap h_{B[h_i^{-1}]}(T)$ with the set of all functions $f : B[h_i^{-1}] \to T$ that map $a = a_i/h_i$ to V, which is the basic open subset $U_{a_i/h_i,V}$ of $h_{B[h_i^{-1}]}(T)$. This shows that $\Theta(U_{a,V})$ is open in $h_B(T)$ and completes the proof of (F1).

Note that thanks to (F1), (A3) immediately implies (F3). Property (A2) implies (F2) for limits of affine ordered blue k-schemes.

We continue with the proof of (F2) for arbitrary limits. If $X = \lim \mathcal{D}$ for a diagram $\mathcal{D}$ of ordered blue schemes, then the canonical projections $\pi_Z : X \to Z$ to the objects Z of $\mathcal{D}$ induce continuous maps $\pi_{Z,T} : X(T) \to Z(T)$, which induce the canonical bijection $\Psi : X(T) \to \lim\big(\mathcal{D}(T)\big)$, which is thus continuous. We are left with showing that Ψ is open. Since every limit is an equalizer of products, we can restrict ourselves to the treatment of these particular types of limits. We will demonstrate the proof for fibre products, which includes equalizers and finite products, and leave the case of infinite products, whose proof is analogous, to the reader.

Consider morphisms $X \to Z \leftarrow Y$ and the fibre product $X \times_Z Y$. For an open subset $\widetilde{W}$ of $X \times_Z Y(T)$, we have to show that $W = \Psi(\widetilde{W})$ is open in $X(T) \times_{Z(T)} Y(T)$. Since $X(T) \times_{Z(T)} Y(T)$ carries the subspace topology of $X(T) \times Y(T)$, a basis open is of the form $W_X \times W_Y$ with opens W_X of $X(T)$ and W_Y of $Y(T)$. By definition, W_X is open in $X(T)$ if and only if $\alpha_T^{-1}(W_X)$ is open in $U(T)$ for all morphisms $\alpha : U \to X$ where U is affine, and W_Y is open in $Y(T)$ if and only if $\beta_T^{-1}(W_Y)$ is open in $V(T)$ for all morphisms $\beta : V \to Y$ where V is affine. Thus $W_X \times W_Y$ is open in $X(T) \times_{Z(T)} Y(T)$ if and only if $(\alpha_T, \beta_T)^{-1}(W_X \times W_Y)$ is open in $U(T) \times V(T)$ for all morphisms $(\alpha, \beta) : U \times V \to X \times_Z Y$ where $U \times V$ is affine.

This shows that W is open in $X(T)\times_{Z(T)}Y(T)$ if and only if $(\alpha_T,\beta_T)^{-1}(W)=(\alpha,\beta)_T^{-1}(\widetilde{W})$ is open in $U(T)\times V(T)=U\times V(T)$ for all $(\alpha,\beta):U\times V\to X\times_Z Y$ where $U\times V$ is affine. This follows from the openness of $\widetilde{W}$. Thus Ψ is open. This completes the proof of (F2).

We turn to the proof of (F4). Let $\iota:U\to X$ be an open immersion. By functoriality, the inclusion $\iota_T:U(T)\to X(T)$ is continuous. We are left with showing that the ι_T is open.

Let $W\subset U(T)$ be an open subset and $W'=\iota_T(W)$ its image in $X(T)$. Then W' is open if and only if $Z=\alpha_T^{-1}(W')$ is open in $h_B(T)$ for every ordered blueprint B and every morphism $\alpha:V\to X$ where $V=\operatorname{Spec}B$. The base change of $\iota:U\to X$ along $\alpha:V\to X$ yields the open immersion $\psi:Z\to V$ where $Z=U\times_X V$. Since $W'=\iota_T(W)$ is in the image of ι_T, the subset Z of $V(T)$ is contained in the image of ψ_T.

We can cover V' by principal opens V_i' of V, i.e. the restriction $\psi_i:V_i'\to V$ of $\psi:V'\to V$ is induced by a morphism $f_i:B\to B[h_i^{-1}]$ of ordered blueprints for each i. By (F1), the canonical bijection $\operatorname{Spec}V_i'\to h_{B[h_i^{-1}]}(T)$ is a homeomorphism and by (A4), the induced map $h_{B[h_i^{-1}]}(T)\to h_B(T)$ is an open topological embedding. We conclude that Z is open in $V(T)$ if $Z_i=\psi_{i,T}^{-1}(Z)$ is open in $h_{B[h_i^{-1}]}(T)$ for every i. But $Z_i=\beta_{i,T}^{-1}(W)$ where β_i is the composition of the inclusion $V_i'\to V'$ with the canonical projection $V'=U\times_X V\to U$. By the definition of the fine topology for $U(T)$, $\beta_{i,T}^{-1}(W)$ is open in $V_i'(T)$. This completes the proof of (F4).

We turn to the proof of (F5). Let $X=\bigcup U_i$ be a covering of open subschemes. Since T is local, we have an equality $X(T)=\bigcup U_i(T)$ of sets. By (F4), this is indeed a covering by open subspaces. Thus (F5).

We turn to the proof of (F6). Since the property of being a topological embedding is a local property, we can assume that X is affine. Since closed immersions are affine morphisms by definition, the closed subscheme Y of X is also affine and the closed immersion $Y\to X$ is induced by a surjection $B\to C$ of ordered blue k-algebras. By (F1), we do not have to worry about whether B and C are global, and thus (F6) follows from its analogue (A6) for the affine topology. This completes the proof of (F6).

We turn to the proof of (F7), assuming that T is a topological semiring. Since being a homeomorphism is a local property, we can restrict ourselves to the affine case, i.e. $X=\operatorname{Spec}B$ and $X^+=\operatorname{Spec}B^+$. By (F1), we know that $X(T)=h_B(T)$ and $X^+(T)=h_{B^+}(T)$ as topological spaces, so it suffices to study the latter spaces. By functoriality, the bijection $\Xi:h_{B^+}(T)\to h_B(T)$ that is induced by the morphisms $B\to B^+$ is continuous.

For verifying that Ξ is open, consider a basic open $U_{\sum a_i,V}$ where $\sum_{i=1}^n a_i$ is an element of B^+, with $a_i\in B$, and $V\subset T$ is open. Then

$$\Xi(U_{\sum a_i,V}) \;=\; \{\,f:B\to T\,|\,\sum f(a_i)\in V\,\} \;=\; \{\,f:B\to T\,|\,\big(f(a_1),\dots,f(a_n)\big)\in\alpha_n^{-1}(V)\,\}$$

where $\alpha_n:T^n\to T$ is the n-fold addition, which is a continuous map since T is a topological semiring. Thus $\alpha_n^{-1}(V)$ is an open subset of T^n, which can be covered by basic opens of the form $V_{k,1}\times\cdots\times V_{k,n}$ where k ranges through some index set I. Thus

$$\Xi(U_{\sum a_i,V}) \;=\; \bigcup_{k\in I} U_{a_1,V_{k,1}}\cap\cdots\cap U_{a_n,V_{k,n}}$$

is an open subset of $h_B(T)$. This completes the proof of (F7).

We turn to the proof of (F8), assuming that T is a topological semiring and Hausdorff. Since being a closed topological embedding is a local property, we can assume that $X=\operatorname{Spec}B$ and $Y=\operatorname{Spec}C$ are affine, and that the closed immersion $Y\to X$ is induced by a surjection $f:B\to C$ of ordered blue k-algebras. By (F6), we know already that the inclusion $Y(T)\to X(T)$ is a topological embedding. By (F1), the bijections $X(T)\to h_B(T)$ and $Y(T)\to h_C(T)$ are homeomorphisms. Therefore we are left with showing that the image of $\iota_T:h_C(T)\to h_B(T)$ is a closed subset of $h_B(T)$

Since $f : B \to C$ is surjective, we have $C = B /\!\!/ \mathcal{R}$ for some subaddition $\mathcal{R}$ on $B^\bullet$, which is a relation on $\mathbb{N}[B^\bullet]$. Thus

$$\iota_T(h_C(T)) \;=\; \{\, f : B \to T \,|\, \sum f(a_i) = \sum f(b_j) \text{ for all } (\sum a_i, \sum b_j) \in \mathcal{R} \,\}.$$

The condition $\sum f(a_i) = \sum f(b_j)$ can be rewritten as $\alpha_n\big(f(a_1),\dots,f(a_n)\big) = \alpha_m\big(f(b_1),\dots,f(b_m)\big)$ where $\alpha_n : T^n \to T$ denotes the n-fold addition, which is continuous since T is a topological semiring. This is, in turn, equivalent to $\big(f(a_1),\dots,f(a_n), f(b_1),\dots,f(b_m)\big) \in (\alpha_n,\alpha_m)^{-1}(\Delta)$ where Δ is the diagonal of $T \times T$. Since T is Hausdorff, Δ is a closed subset of $T \times T$, and therefore $\Delta' = (\alpha_n,\alpha_m)^{-1}(\Delta)$ is a closed subset of $T^n \times T^m$. This means that

$$\Delta' \;=\; \bigcap_{k\in I} V_{k,1} \times \cdots \times V_{k,n+m}$$

for certain closed subsets $V_{k,l}$ of T where k varies through some index set I and $l = 1,\dots,n+m$. We conclude that

$$\iota_T(h_C(T)) \;=\; \bigcap_{k\in I} U_{a_1,V_{k,1}} \cup \cdots \cup U_{a_n,V_{k,n}} \cup U_{b_1,V_{k,n+1}} \cup \cdots \cup U_{b_m,V_{k,n+m}}$$

This shows that $\iota_T(h_C(T))$ is a closed subset of $h_B(T)$, which completes the proof of the theorem. □

6. Connection to Toën and Vaquié's relative schemes

In this section, we explain the differences of the scheme theory used in this text to the viewpoints employed in an earlier version of this text and in the papers [21] and [22] of Jeff and Noah Giansiracusa.

These latter approaches are based on Toën and Vaquié's paper [51], which generalizes the concept of a scheme to a fairly broad context. Roughly speaking, this can be done for any category $\mathcal{A}$ of algebraic structures with a sufficiently well-behaved category $\mathcal{M}$ of modules. The modules provide a Grothendieck pretopology for $\mathcal{A}$, which allows us to define a *scheme relative to* $\mathcal{M}$ as a locally representable sheaf on $\mathcal{A}$.

Toën and Vaquié's theory applies, in particular, to the category of ordered blueprints and to the category of semirings. It turns out that the corresponding categories of schemes differ from the categories $\mathrm{Sch}_{\mathbb{F}_1}$ and $\mathrm{Sch}_{\mathbb{N}}^+$ that we use in this text.

To avoid confusion between these different concepts of schemes for ordered blueprints and semirings, we will call the objects of $\mathrm{Sch}_{\mathbb{F}_1}$ and $\mathrm{Sch}_{\mathbb{N}}^+$ *geometric ordered blue schemes* and *geometric semiring schemes*, respectively, while use the expressions *subcanonical ordered blue schemes* and *subcanonical semiring schemes* to refer to the corresponding objects in Toën and Vaquié's theory.

Thanks to results in [31] and [39], we are able to bypass the technically involved definition from [51] by giving an explicit description of subcanonical ordered blue schemes and subcanonical semiring schemes as ordered blueprinted spaces.

Note that the results in [31] only cover algebraic blueprints. The more general statements for ordered blueprints, as described below, can be proven in complete analogy to [31]. Alternatively, one can use the bijection $B^{\mathrm{core}} \to B$ to deduce the claims for ordered blueprints B from the corresponding results for its algebraic core B^{core}.

6.1. Subcanonical ordered blue schemes. Let B be an ordered blueprint. We define the *subcanonical spectrum* $\mathrm{Spec}^{\mathrm{can}} B$ of B as the following blueprinted space. The underlying set of $\mathrm{Spec}^{\mathrm{can}} B$ is that of $\mathrm{Spec}\, B^\bullet$, i.e. the points of $\mathrm{Spec}^{\mathrm{can}} B$ are the subsets $\mathfrak{p}$ of B with $0 \in \mathfrak{p}$ and $\mathfrak{p}B = \mathfrak{p}$ such that $S = B - \mathfrak{p}$ is a multiplicative subset of B. The topology of $\mathrm{Spec}^{\mathrm{can}} B$ is generated by the principal opens

$$U_h \;=\; \{\, \mathfrak{p} \in \mathrm{Spec}^{\mathrm{can}} B \,|\, h \notin \mathfrak{p} \,\}$$

for $h \in B$. The structure sheaf $\mathcal{O}_X$ of $X = \operatorname{Spec}^{\mathrm{can}} B$ is characterized by its values $\mathcal{O}_X(U_h) = B[h^{-1}]$ on principal opens.

In particular, we have $\mathcal{O}_X(X) = B$ for all ordered blueprints B, in contrast to the negative result for the geometric spectrum $\operatorname{Spec} B$. This implies that the contravariant functor $\operatorname{Spec}^{\mathrm{can}} : \mathrm{OBlpr} \to \mathrm{OBlprSp}$ is fully faithful. We call objects in the essential image of this functor affine subcanonical ordered blue schemes.

A *subcanonical ordered blue scheme* is a blueprinted space that is covered by affine subcanonical ordered blue schemes. This defines the full subcategory $\mathrm{Sch}_{\mathbb{F}_1}^{\mathrm{can}}$ of subcanonical ordered blue schemes in OBlprSp.

The analogue of Corollary 4.4 holds in this context, but with $\Gamma\mathrm{OBlpr}$ replaced by OBlpr. In particular, the global section functor $\Gamma^{\mathrm{can}} : \mathrm{Sch}_{\mathbb{F}_1}^{\mathrm{can}} \to \mathrm{OBlpr}$ is a right inverse to $\operatorname{Spec}^{\mathrm{can}} : \mathrm{OBlpr} \to \mathrm{Sch}_{\mathbb{F}_1}^{\mathrm{can}}$ and an adjoint in the sense that $\operatorname{Hom}(X, \operatorname{Spec}^{\mathrm{can}} B) = \operatorname{Hom}(B, \Gamma^{\mathrm{can}} X)$ for all subcanonical ordered blue schemes X and ordered blueprints B.

To conclude, we comment on the following remarkable feature of subcanonical ordered blue schemes. Since the underlying monoid $B^\bullet$ of an ordered blueprint B has a unique maximal k-ideal, which is $\mathfrak{m} = B - B^\times$, the subcanonical spectrum $\operatorname{Spec}^{\mathrm{can}} B$ has a unique closed point. As a consequence, every closed point x of an ordered blue scheme X has a unique open affine neighbourhood U, which is isomorphic to $\operatorname{Spec}^{\mathrm{can}} \mathcal{O}_{X,x}$.

6.2. Subcanonical semiring schemes. Let B be a semiring. An *ideal of* B is an additively closed subset I of B with $0 \in I$ and $IB = I$. A *prime ideal of* B is an ideal $\mathfrak{p}$ of B such that $S = B - \mathfrak{p}$ is a multiplicative subset. Note that every k-ideal is an ideal, but that in general, not every ideal is a k-ideal. For instance, the ideal $\mathfrak{m} = \mathbb{N} - \{1\}$ of $\mathbb{N}$ is not a k-ideal.

We define the *subcanonical spectrum* $\operatorname{Spec}^{+,\mathrm{can}} B$ of B as the following blueprinted space. The underlying topological space of $\operatorname{Spec}^{+,\mathrm{can}} B$ consists of the prime ideals of B, and the topology of $\operatorname{Spec}^{+,\mathrm{can}} B$ is generated by the principal opens

$$U_h \; = \; \{\mathfrak{p} \in \operatorname{Spec}^{+,\mathrm{can}} B \,|\, h \notin \mathfrak{p}\}$$

for $h \in B$. The structure sheaf $\mathcal{O}_X$ of $X = \operatorname{Spec}^{+,\mathrm{can}} B$ is characterized by its values $\mathcal{O}_X(U_h) = B[h^{-1}]$ on principal opens.

In particular, we have $\mathcal{O}_X(X) = B$ for all ordered blueprints B, in contrast to the negative result for the geometric spectrum $\operatorname{Spec} B$. This implies that the contravariant functor $\operatorname{Spec}^{+,\mathrm{can}} : \mathrm{OBlpr} \to \mathrm{OBlprSp}$ is fully faithful. We call objects in the essential image of this functor affine subcanonical semiring schemes.

A *subcanonical semiring scheme* is a blueprinted space that is covered by affine subcanonical semiring schemes. This defines the full subcategory $\mathrm{Sch}_{\mathbb{N}}^{+,\mathrm{can}}$ of subcanonical ordered blue schemes in OBlprSp.

Similar to the case of subcanonical ordered blue schemes, the global section functor $\Gamma^{+,\mathrm{can}} : \mathrm{Sch}_{\mathbb{N}}^{+,\mathrm{can}} \to \mathrm{SRings}$ is a left inverse and an adjoint to $\operatorname{Spec}^{+,\mathrm{can}} : \mathrm{SRings} \to \mathrm{Sch}_{\mathbb{N}}^{+,\mathrm{can}}$ in the sense that $\operatorname{Hom}(X, \operatorname{Spec}^{+,\mathrm{can}} B) = \operatorname{Hom}(B, \Gamma^{\mathrm{can}} X)$ for all subcanonical semiring schemes X and all semirings B.

The category $\mathrm{Sch}_{\mathbb{Z}}^{+}$ of usual schemes is naturally identified with the full subcategory of all subcanonical semiring schemes whose structure sheaves take values in Rings. The embedding $\iota : \mathrm{Sch}_{\mathbb{Z}}^{+} \to \mathrm{Sch}_{\mathbb{N}}^{+,\mathrm{can}}$ comes with a right adjoint and right inverse $- \otimes_{\mathbb{N}} \mathbb{Z} : \mathrm{Sch}_{\mathbb{N}}^{+,\mathrm{can}} \to \mathrm{Sch}_{\mathbb{Z}}^{+}$ that sends the subcanonical spectrum $\operatorname{Spec}^{+,\mathrm{can}} B$ of a semiring B to the spectrum $\operatorname{Spec} B_{\mathbb{Z}}$ of the ring $B_{\mathbb{Z}} = B \otimes_{\mathbb{N}} \mathbb{Z}$.

The functor $(-)^+ : \mathrm{OBlpr} \to \mathrm{SRings}$ extends naturally to a functor $(-)^+ : \mathrm{Sch}_{\mathbb{F}_1}^{\mathrm{can}} \to \mathrm{Sch}_{\mathbb{N}}^{+,\mathrm{can}}$ that sends the subcanonical spectrum $\operatorname{Spec}^{+,\mathrm{can}} B$ of an ordered blueprint B to $\operatorname{Spec}^{+,\mathrm{can}} B^+$. In contrast to the analogous situation for geometric blue schemes, the inclusion $\mathrm{SRings} \to \mathrm{OBlpr}$

fails to extend to a functor from $\mathrm{Sch}_{\mathbb{N}}^{+,\mathrm{can}}$ to $\mathrm{Sch}_{\mathbb{F}_1}^{\mathrm{can}}$ since the subcanonical Zariski topology on SRings is strictly stronger than the restriction of the subcanonical Zariski topology on OBlpr.

6.3. Comparison of geometric and subcanonical schemes. Since every prime k-ideal of an ordered blueprint B is a prime k-ideal of its underlying monoid $B^\bullet$, the spectrum $\operatorname{Spec} B$ appears naturally as a subspace of the subcanonical spectrum $\operatorname{Spec}^{\mathrm{can}} B$. As a consequence, the restriction from $\operatorname{Spec}^{\mathrm{can}} B$ to $\operatorname{Spec} B$ defines a functor from subcanonical to geometric spectra, and this functor extends to a functor $\mathcal{G} : \mathrm{Sch}_{\mathbb{F}_1}^{\mathrm{can}} \to \mathrm{Sch}_{\mathbb{F}_1}$.

Similarly, every prime k-ideal of a semiring B is a prime ideal, and the spectrum $\operatorname{Spec} B$ appears naturally as a subspace of the subcanonical spectrum $\operatorname{Spec}^{+,\mathrm{can}} B$. This extends to a functor $\mathcal{G}^+ : \mathrm{Sch}_{\mathbb{N}}^{+,\mathrm{can}} \to \mathrm{Sch}_{\mathbb{N}}^{+}$.

Since not every covering of $\operatorname{Spec} B$ comes from a covering of $\operatorname{Spec}^{\mathrm{can}} B$, it is not possible to extend the association that maps $\operatorname{Spec} B$ to $\operatorname{Spec}^{\mathrm{can}} B$ to a functor from $\mathrm{Sch}_{\mathbb{F}_1}$ to $\mathrm{Sch}_{\mathbb{F}_1}^{\mathrm{can}}$, even if B is a global ordered blueprint. This works, however, for the subclass of geometric ordered blue schemes that can be covered by the spectra of local blueprints such that their pairwise intersections are principal opens. This yields a functor $\mathcal{F}$ from the corresponding full subcategory of $\mathrm{Sch}_{\mathbb{F}_1}$ to $\mathrm{Sch}_{\mathbb{F}_1}^{\mathrm{can}}$.

The following collection of results is Theorem A in [31], bearing in mind that we extend these results from algebraic blueprints to ordered blueprints.

Theorem 6.1. *The diagram of functors*

$$\begin{tikzcd}
\mathrm{Sch}_{\mathbb{F}_1}^{\mathrm{can}} \arrow[rr, "\mathcal{G}"] \arrow[d, "(-)^+"] & & \mathrm{Sch}_{\mathbb{F}_1} \arrow[ll, dashed, bend left, "\mathcal{F}"] \arrow[d, shift left, "(-)^+"] \\
\mathrm{Sch}_{\mathbb{N}}^{+,\mathrm{can}} \arrow[rr, "\mathcal{G}^+"] \arrow[dr, shift right, "-\otimes_{\mathbb{N}}\mathbb{Z}"'] & & \mathrm{Sch}_{\mathbb{N}}^{+} \arrow[u, shift left, "\iota"] \arrow[dl, shift left, "-\otimes_{\mathbb{N}}\mathbb{Z}"] \\
& \mathrm{Sch}_{\mathbb{Z}}^{+} \arrow[ul, shift right, "\iota"'] \arrow[ur, shift left, "\iota"] &
\end{tikzcd}$$

satisfies the following properties:

(i) *the outer square and both the inner and the outer triangle commute;*
(ii) *the embeddings ι are left inverse and right adjoint to the respective base extension functors $(-)^+$ and $-\otimes_{\mathbb{N}} \mathbb{Z}$;*
(iii) *the functor $\mathcal{G}$ is induced by the identity functor on* Blpr *and $\mathcal{G}^+$ is induced by the identity functor on* SRings*;*
(iv) *$\mathcal{F}$ is a partially defined right inverse functor to $\mathcal{G}$ whose domain includes all monoidal schemes and all geometric blue schemes that are locally of finite type over a blue field.*

6.4. Relevance for scheme theoretic tropicalization. Finally, we have reached the point that we can explain the differences between the different theories of tropicalization, as described in [21] and [22], in an earlier version of this paper and in the current version.

For making sense of the analytification of a k-scheme as a "universal tropicalization", we require an extension of the embedding Rings $\to$ OBlpr to the respective categories of schemes. This works in the geometric setting where we have a functor $\iota : \mathrm{Sch}_{\mathbb{Z}}^{+} \to \mathrm{Sch}_{\mathbb{F}_1}$, but this embedding fails to extend to a functor from $\mathrm{Sch}_{\mathbb{Z}}^{+}$ to $\mathrm{Sch}_{\mathbb{F}_1}^{\mathrm{can}}$. Therefore we employ the approach via geometric blue schemes in this paper.

In contrast to this digression concerning the universal tropicalization, the choice of using geometric blue schemes for "tropicalizations with coordinates" is purely a matter of exposition. The use of two different theories of ordered blue schemes would complicate the formulation of our theorems in a technical way. This choice comes with the cost that we have to assume that

the domain k of the valuation $v : k \to T$ is with -1, in contrast to the corresponding results for subcanonical ordered blue schemes, as used in an earlier version of this text. To confirm, the reader should rest assured that the results concerning tropicalizations are correct in this earlier version and equivalent to the corresponding results in this paper in the following precise sense, assuming that k is with -1.

Applying the functor $\mathcal{G} : \mathrm{Sch}_{\mathbb{F}_1}^{\mathrm{can}} \to \mathrm{Sch}_{\mathbb{F}_1}$ to the tropicalization of a k-scheme X as a subcanonical ordered blue scheme yields the tropicalization of X as a geometric blue scheme, as considered in this paper.

Conversely, if X is locally of finite type over a field k, then its tropicalization as a geometric ordered blue scheme is locally of finite type over the value group of the valuation of k. This means that it lies in the domain of the partial right inverse $\mathcal{F}$ of $\mathcal{G}$, and its image under $\mathcal{F}$ is the tropicalization of X as a subcanonical ordered blue scheme.

Part 2. Tropicalization

In the second part of the paper, we give our definition of a tropicalization $\mathrm{Trop}_v(X)$ as a solution to the moduli problem of extensions of a given valuation $v : k \to T$ to a given ordered blue k-scheme X with values in ordered blueprints over T. Our central results on tropicalizations show their existence for totally positive and for idempotent T. In both cases, our constructions of tropicalizations pass through a base change along v. In the latter case, we gain an alternative description in terms of a generalization of the Giansiracusa bend relation to ordered blueprints.

In the subsequent sections, we show how our definition of tropicalization recovers and improves other concepts to tropicalization and analytification. In subsequent sections, we address Berkovich analytification, Kajiwara-Payne tropicalization, Foster-Ranganathan tropicalization, Giansiracusa tropicalization, Maclagan-Rincón weights, Macpherson analytification, Thuillier analytification and Ulirsch tropicalization.

7. Scheme theoretic tropicalization

Let k and T be ordered blueprints, $v : k \to T$ a valuation and X an ordered blue k-scheme. In this section, we introduce the functor $\mathrm{Val}_v(X,-)$ of valuations of X over v. We define a tropicalization of X along v as a ordered blue T-scheme that represents $\mathrm{Val}_v(X,-)$ and construct tropicalizations if T is totally positive or idempotent.

7.1. Tropicalization as a moduli space. Let k be an ordered blueprint. We denote the category of ordered blue k-algebras together with k-linear morphisms by $\mathrm{Alg}_k^{\mathrm{ob}}$.

Let $v : k \to T$ be a valuation, i.e. a morphism $v^\bullet : k^\bullet \to T^\bullet$ between the underlying monoids of k and T together with a morphism $\tilde{v} : k^{\mathrm{mon}} \to T^{\mathrm{pos}}$ such that the diagram

$$\begin{tikzcd} k^\bullet \arrow[r, "v^\bullet"] \arrow[d] & T^\bullet \arrow[d] \\ k^{\mathrm{mon}} \arrow[r, "\tilde{v}"] & T^{\mathrm{pos}} \end{tikzcd}$$

commutes. Let B be an ordered blue k-algebra and S an ordered blue T-algebra. A *valuation $w : B \to S$ over v* is a morphism $w^\bullet : B^\bullet \to S^\bullet$ together with a morphism $\tilde{w} : B^{\mathrm{mon}} \to S^{\mathrm{pos}}$ such that the diagram

$$\begin{tikzcd} k^\bullet \arrow[rr, "v^\bullet"] \arrow[dr] \arrow[dd] & & T^\bullet \arrow[dr] \arrow[dd] & \\ & B^\bullet \arrow[rr, "w^\bullet"] \arrow[dd] & & S^\bullet \arrow[dd] \\ k^{\mathrm{mon}} \arrow[rr, "\tilde{v}"] \arrow[dr] & & T^{\mathrm{pos}} \arrow[dr] & \\ & B^{\mathrm{mon}} \arrow[rr, "\tilde{w}"] & & S^{\mathrm{pos}} \end{tikzcd}$$

commutes. We denote by $\mathrm{Val}_v(B,S)$ the set of valuations $w : B \to S$ over v.

A morphism $f : S \to S'$ of ordered blue T-algebras sends a valuation $w : B \to S$ to the valuation $f \circ w : B \to S'$ where we define $(f \circ w)^\bullet = f^\bullet \circ w^\bullet$ and $(f \circ w)^\sim = f^{\mathrm{pos}} \circ \tilde{w}$. This shows that $\mathrm{Val}_v(B,S)$ is functorial in S, and we obtain the *functor of valuations* $\mathrm{Val}_v(B,-) : \mathrm{Alg}_k^{\mathrm{ob}} \to \mathrm{Alg}_T^{\mathrm{ob}}$ *on B over v*.

Definition 7.1. Let k and T be ordered blueprints, $v : k \to T$ be a valuation and B an ordered blue k-algebra. A *tropicalization of B along v* is an ordered blue T-algebra $\mathrm{Trop}_v(B)$ together with a valuation $w^{\mathrm{univ}} : B \to \mathrm{Trop}_v(B)$ that is universal, i.e. for every valuation $w : B \to S$ over v, there is a unique morphism $f : S \to \mathrm{Trop}_v(B)$ of ordered blue T-algebras such that $w = f \circ w^{\mathrm{univ}}$.

In other words, a tropicalization of B along v is an ordered blue T-algebra that represents the functor $\mathrm{Val}_v(B,-)$, i.e. the fine moduli space of all valuations of B over v. By the Yoneda lemma, the tropicalization of B along v is unique up to unique isomorphism if it exists.

This notion can be geometrized as follows. Let X be an ordered blue scheme over k with structure morphism $X \to \operatorname{Spec} k$ and Y be an ordered blue T-scheme. A *valuation of $\omega : X \to Y$ over v* is a morphism $\omega^\bullet : Y^\bullet \to X^\bullet$ together with a morphism $\tilde{\omega} : Y^{\mathrm{pos}} \to X^{\mathrm{mon}}$ such that the diagram

$$\begin{tikzcd}
 & \operatorname{Spec} T^\bullet \arrow[rr, "(v^\bullet)^*"] & & \operatorname{Spec} k^\bullet \\
Y^\bullet \arrow[ur] \arrow[rr, "\omega^\bullet"] & & X^\bullet \arrow[ur] & \\
 & \operatorname{Spec} T^{\mathrm{pos}} \arrow[uu] \arrow[rr, "\tilde{v}^*"] & & \operatorname{Spec} k^{\mathrm{mon}} \arrow[uu] \\
Y^{\mathrm{pos}} \arrow[uu] \arrow[ur] \arrow[rr, "\tilde{\omega}"] & & X^{\mathrm{mon}} \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

commutes.

Definition 7.2. Let k and T be ordered blueprints, $v : k \to T$ be a valuation and X an ordered blue k-scheme. A *tropicalization of X along v* is an ordered blue T-scheme $\mathrm{Trop}_v(X)$ together with a valuation $\omega^{\mathrm{univ}} : \mathrm{Trop}_v(X) \to X$ that is universal, i.e. for every valuation $\omega : Y \to X$ over v, there is a unique morphism $\varphi : Y \to \mathrm{Trop}_v(X)$ of ordered blue T-schemes such that $\omega = \omega^{\mathrm{univ}} \circ \varphi$.

Like in the affine situation, a tropicalization $\mathrm{Trop}_v(X)$ of X along v represents the *functor of valuations* $\mathrm{Val}_v(X,-) : \mathrm{Sch}_T^{\mathrm{ob}} \to \mathrm{Sets}$ *on X over v*, which sends a T-scheme Y to the set $\mathrm{Val}_v(X,Y)$ of valuations $\omega : X \to Y$ over v.

We call T the *base of* $\mathrm{Trop}_v(X)$ or the *tropicalization base* if the context is clear.

7.2. Tropicalization for a totally positive tropicalization base. Let $v : k \to T$ be a valuation. For an ordered blue k-algebra B, we can make precise the idea that its tropicalization is the base change of B along the valuation $v : k \to T$ in case that T is totally positive.

We begin with the following preliminary observation. Let S be an ordered blue T-algebra and B an ordered blue k-algebra. Then the morphism $\tilde{w} : B^{\mathrm{mon}} \to S^{\mathrm{pos}}$ is uniquely determined by the valuation $w : B \to S$ since $B^{\mathrm{mon}} \to B$ is a bijection. This yields a map

$$\mathrm{Val}_v(B,S) \quad \longrightarrow \quad \mathrm{Hom}_T(B_v,S)$$

where we write B_v for the tensor product $B^{\mathrm{mon}} \otimes_{k^{\mathrm{mon}}} T^{\mathrm{pos}}$.

Lemma 7.3. *The map* $\mathrm{Val}_v(B,S) \to \mathrm{Hom}_T(B_v,S^{\mathrm{pos}})$ *is a bijection if T is idempotent or totally positive.*

Proof. If T is idempotent or totally positive, then every ordered blue T-algebra S is idempotent or totally positive as well. By Lemma 2.15, S is strictly conic, and by Corollary 2.11, the canonical morphism $S \to S^{\mathrm{pos}}$ is a bijection. This shows that a valuation w is determined by $\tilde{w}$ in this case. $\square$

Theorem 7.4. *If S is totally positive, then $\mathrm{Trop}_v(B) = B^{\mathrm{mon}} \otimes_{k^{\mathrm{mon}}} T^{\mathrm{pos}}$ is a tropicalization of B along v.*

Proof. If T is totally positive, then every ordered blue T-algebra S is totally positive as well, i.e. $S = S^{\mathrm{pos}}$. Therefore, the theorem follows at once from Lemma 7.3. □

As explained in section 2.6, we can model partially additive blueprints as monomial blueprints, which includes semirings, blueprints with -1 and monoids. By Corollary 2.13, we can model strictly conic blueprints as totally positive blueprints, which includes $\mathbb{R}_{\geqslant 0}$ and all idempotent semirings.

Since a valuation $v: k \to T$ from a monomial blueprint k to a totally positive blueprint T is nothing else than a morphism of ordered blueprints, we draw the following conclusion.

Corollary 7.5. *Let $v: k \to T$ be a valuation from a monomial blueprint k to a totally positive blueprint T and X a monomial ordered blue k-scheme. Then the base change $X \otimes_k T$ is a tropicalization of X along v.*

Proof. This follows at once from Theorem 7.4 by choosing affine pavings for X. □

Remark 7.6. In the given framework of ordered blue schemes, the functor $(-)^{\mathrm{mon}}: \mathrm{OBlpr} \to \mathrm{OBlpr}$ fails to extend to an endofunctor on $\mathrm{Sch}_{\mathbb{F}_1}$. It does, however extend to an endofunctor $(-)^{\mathrm{mon}}: \mathrm{Sch}_{\mathbb{F}_1}^{\mathrm{can}} \to \mathrm{Sch}_{\mathbb{F}_1}^{\mathrm{mon}}$ of the category of subcanonical ordered blue schemes, as explained in [34]. Therefore, we have $\mathrm{Trop}_v(X) = X^{\mathrm{mon}} \otimes_{k^{\mathrm{mon}}} T$ in $\mathrm{Sch}_{\mathbb{F}_1}^{\mathrm{can}}$ if T is totally positive.

Remark 7.7. Corollary 7.5 makes precise the idea of the tropicalization as a base change along a valuation, as explained in the introduction. As we explained in Remark 2.7, $(-)^{\mathrm{mon}}: \mathrm{Halos} \to \mathrm{OBlpr}^{\mathrm{mon}}$ is a fully faithful embedding of the category of halos, which are ordered semirings together with subadditive morphisms, into the category of monomial blueprints. In the following, we identify halos with their images in $\mathrm{OBlpr}^{\mathrm{mon}}$.

Let $k \to B$ and $v: k \to T$ be a morphism of halos and assume that T is totally positive, e.g. $T = (\mathbb{R}_{\geqslant 0}^{\mathrm{pos}})^{\mathrm{mon}}$ or $T = (\mathbb{T}^{\mathrm{pos}})^{\mathrm{mon}}$. Then the tensor product $B \otimes_k T$ exists in $\mathrm{OBlpr}^{\mathrm{mon}}$ and it is a tropicalization of B along v.

Note that the tensor product $B \otimes_k T$ is totally positive, but that it is typically not a halo. At the moment of writing, it is not clear to me in which situations a tensor product of halos exists in Halos.

7.3. The bend functor. In this section, we introduce the bend relation, which generalizes the corresponding concept from [21] to our setting; see section 11 for details on the connection to [21].

Definition 7.8. Let $v: k \to T$ be a valuation and B an ordered blue k-algebra. The *bend of B along v* is the ordered blue T-algebra

$$\mathrm{Bend}_v(B) \quad = \quad B^{\bullet} \otimes_{k^{\bullet}} T /\!\!/ \mathrm{bend}_v(B)$$

whose subaddition is generated by the subaddition of T and the *bend relation*

$$\mathrm{bend}_v(B) \quad = \quad \Big\langle\, a \otimes 1 + \sum b_j \otimes 1 \equiv \sum b_j \otimes 1 \,\Big|\, a \leqslant \sum b_j \text{ in } B \,\Big\rangle.$$

A morphism $f: B \to C$ of ordered blue k-algebras defines a map

$$\begin{array}{cccc} \mathrm{Bend}_v(f): & \mathrm{Bend}_v(B) & \longrightarrow & \mathrm{Bend}_v(C). \\ & b \otimes t & \longmapsto & f(b) \otimes t \end{array}$$

This map is clearly multiplicative and T-linear. The bend relations are preserved for the following reason. If a relation $a \leqslant \sum b_j$ in B induces the relation $a \otimes 1 + \sum b_j \otimes 1 \equiv \sum b_j \otimes 1$ on

$\mathrm{Bend}_v(B)$, then the image relation $f(a) \leqslant \sum f(b_j)$ in C induces the relation $f(a)\otimes 1 + \sum f(b_j)\otimes 1 \equiv \sum f(b_j)\otimes 1$ on $\mathrm{Bend}_v(C)$. This defines the *bend functor*

$$\mathrm{Bend}_v : \mathrm{Alg}_k^{\mathrm{ob}} \longrightarrow \mathrm{Alg}_T^{\mathrm{ob}}.$$

Note that the bend of B along v is idempotent since $1 \leqslant 1$ in B implies $1\otimes 1 + 1\otimes 1 \equiv 1\otimes 1$ in $\mathrm{bend}_v(B)$. Since the bend of B depends only on relations of the form $a \leqslant \sum b_j$, we have $\mathrm{Bend}_v(B^{\mathrm{mon}}) = \mathrm{Bend}_v(B)$. By the very definition of $\mathrm{bend}_v(B)$, the bend of B is algebraic if T is so.

Lemma 7.9. *Let B be an ordered blue k-algebra and $S \subset B$ a multiplicative subset. Let $S_v = S\otimes\{1\}$ be its image in* $\mathrm{Bend}_v(B)$. *Then the association $\frac{a\otimes b}{s} \mapsto \frac{a}{s}\otimes b$ defines a canonical isomorphism $S_v^{-1}\mathrm{Bend}_v(B) \to \mathrm{Bend}_v(S^{-1}B)$.*

Proof. The canonical morphism $S_v^{-1}\mathrm{Bend}_v(B) \to \mathrm{Bend}_v(S^{-1}B)$ is induced from the isomorphism of ordered blue T-algebras

$$S_v^{-1}\big(B^\bullet \otimes_{k^\bullet} T\big) \quad\longrightarrow\quad \big(S^{-1}B^\bullet \otimes_{k^\bullet} T\big).$$

We have to show that this morphism of monoids identifies the respective subadditions of $S_v^{-1}\mathrm{bend}_v(B)$ and $\mathrm{bend}_v(S^{-1}B)$.

The generators for the bend relations of $S_v^{-1}\mathrm{Bend}_v(B)$ and $\mathrm{Bend}_v(S^{-1}B)$ are of the respective forms

$$\frac{a\otimes 1}{s\otimes 1} + \sum \frac{b_j\otimes 1}{s\otimes 1} \equiv \sum\frac{b_j\otimes 1}{s\otimes 1} \qquad \text{and} \qquad \frac{a}{s_a}\otimes 1 + \sum\frac{b_j}{s_j}\otimes 1 \equiv \sum \frac{b_j}{s_j}\otimes 1$$

where $s, s_a, s_j \in S$ and $a \leqslant \sum b_j$ in B. Note that we use that the subaddition of $S^{-1}B$ is generated by the subaddition of B. Relations of the former type are relations of the latter type, which implies that the canonical map $S_v^{-1}\mathrm{Bend}_v(B) \to \mathrm{Bend}_v(S^{-1}B)$ is a morphism of ordered blueprints. Conversely, every relation of the latter form can be rewritten as

$$\frac{s^{(a)}a\otimes 1}{s\otimes 1} + \sum\frac{s^{(j)}b_j\otimes 1}{s\otimes 1} \quad\equiv\quad \sum\frac{s^{(j)}b_j\otimes 1}{s\otimes 1}$$

where $s = s_a\cdot\prod s_j$, $s^{(a)} = \prod s_j$ and $s^{(j)} = s_a\cdot\prod_{i\neq j} s_i$. □

Lemma 7.10. *Let $v: k\to T$ be a valuation. The functor* $\mathrm{Bend}_v : \mathrm{Alg}_k^{\mathrm{ob}} \to \mathrm{Alg}_T^{\mathrm{ob}}$ *commutes with non-empty colimits.*

Proof. The statement follows if we can show that Bend_v commutes with cofibre products and non-empty coproducts.

Let $C \leftarrow B \to D$ be a diagram of k-algebras. The cofibre product of C and D over B is the tensor product $C\otimes_B D$. We have $(C\otimes_B D)^\bullet = C^\bullet \otimes_{B^\bullet} D^\bullet$ and

$$\mathrm{Bend}_v(C\otimes_B D) \;=\; C^\bullet \otimes_{B^\bullet} D^\bullet \otimes_{k^\bullet} T /\!\!/ \mathrm{bend}_v(C\otimes_B D).$$

A relation $a\otimes b \leqslant \sum c_i\otimes d_i$ in $C\otimes_B D$ must be the sum of a relation of the form $a\otimes 1 \leqslant \sum c_i\otimes 1$ or $1\otimes b \leqslant \sum 1\otimes d_j$ with relations of the form $0 \leqslant \sum c_j'\otimes d_j'$. Since addition of relations of the latter type do not contribute to $\mathrm{bend}_v(C\otimes_B D)$, we can restrict our attention to relations of the former types. If we write

$$\mathrm{bend}_v(C)\otimes 1 \;=\; \Big\langle a\otimes 1\otimes 1 + \sum c_i\otimes 1\otimes 1 \equiv \sum c_i\otimes 1\otimes 1 \,\Big|\, a \leqslant \sum c_i \text{ in } C\Big\rangle$$

and

$$1\otimes\mathrm{bend}_v(D) \;=\; \Big\langle 1\otimes b\otimes 1 + \sum 1\otimes d_j\otimes 1 \equiv \sum 1\otimes d_j\otimes 1 \,\Big|\, b \leqslant \sum d_j \text{ in } D\Big\rangle,$$

then we conclude that $\mathrm{bend}_v(C\otimes_B D) = \langle \mathrm{bend}_v(C)\otimes 1, 1\otimes \mathrm{bend}_v(D)\rangle$. Since

$$C^\bullet \otimes_{B^\bullet} D^\bullet \otimes_{k^\bullet} T \;=\; \big(C^\bullet \otimes_{k^\bullet} T\big) \otimes_{(B^\bullet \otimes_{k^\bullet} T)} \big(D^\bullet \otimes_{k^\bullet} T\big),$$

we conclude that

$$\begin{aligned} C^\bullet \otimes_{B^\bullet} D^\bullet \otimes_{k^\bullet} T /\!\!/ &\langle \mathrm{bend}_v(C)\otimes 1, 1\otimes \mathrm{bend}_v(D)\rangle \\ &= \big(C^\bullet \otimes_{k^\bullet} T /\!\!/ \mathrm{bend}_v(B)\big) \otimes_{(B^\bullet \otimes_{k^\bullet} T /\!\!/ \mathrm{bend}_v(B))} \big(D^\bullet \otimes_{k^\bullet} T /\!\!/ \mathrm{bend}_v(D)\big), \end{aligned}$$

which is $\mathrm{Bend}_v(C)\otimes_{\mathrm{Bend}_v(B)} \mathrm{Bend}_v(D)$ as desired.

Since the coproduct of a non-empty family $\{B_i\}$ of ordered blue k-algebras is represented by the (possibly infinite) tensor product $\bigotimes_k B_i$ over k, it follows from the same argument as for the cofibre product that Bend_v commutes with non-empty coproducts. This completes the proof of the lemma. □

Remark 7.11. Since $\mathrm{Bend}_v(k)$ is idempotent, it is clear that Bend_v does not preserve initial objects if T is not idempotent. If, however, T is idempotent, then it can be proven that $\mathrm{Bend}_v(k)$ is isomorphic to T. It follows that Bend_v preserves all colimits if T is idempotent.

Lemma 7.12. *Let $v : k \to T$ be a valuation and assume that k is with -1. Then Bend_v preserves covering families.*

Proof. Assume that $Y = \mathrm{Spec}\, B$ is covered by principal open subsets $U_{h_i} = \mathrm{Spec}\, B[h_i^{-1}]$. As explained in the proof of Lemma 4.9, the U_{h_i} cover X if and only if 1 is contained in the k-ideal generated by the h_i, i.e. there exists a relation of the form $1+\sum b_k h_{i_k} \equiv \sum c_l h_{i_l}$ with $b_k, c_l \in B$.

Since k, and therefore also B, is with -1, we can add $\sum(-b_k)h_{i_k}$ to the above equation. After renaming the coefficients, we find a relation of the form $1 = \sum b_k h_{i_k}$ with $b_k \in B$.

Applying the bend functor to B and the $B[h_i^{-1}]$ yields the T-scheme $X_T = \mathrm{Spec}\,\mathrm{Bend}_v(B)$ and the principal opens $U_{h\otimes 1} = \mathrm{Bend}_v(U_{h_i}) = \mathrm{Spec}\big(\mathrm{Bend}_v(B)[(h_i\otimes 1)^{-1}]\big)$, using Lemma 7.9. The relation $1 = \sum b_k h_{i_k}$ yields the bend relation $1\otimes 1 + \sum b_k h_{i_k} \otimes 1 = \sum b_k h_{i_k} \otimes 1$ in $\mathrm{Bend}_v(B)$. This shows that $1 = 1\otimes 1$ is contained in the k-ideal generated by the $h_i \otimes 1$. Thus X_T is covered by the $U_{h\otimes 1}$, which completes the proof of the lemma. □

Theorem 7.13. *Let $v : k \to T$ be a valuation and assume that k is with -1. Then the bend functor extends to a functor*

$$\mathrm{Bend}_v : \mathrm{Sch}_k \to \mathrm{Sch}_T$$

that sends an affine open $U = \mathrm{Spec}\, B$ of an ordered blue k-scheme X to the affine open $\mathrm{Bend}_v(U) = \mathrm{Spec}\big(\mathrm{Bend}_v(B)\big)$ *of the ordered blue T-scheme* $\mathrm{Bend}_v(X)$.

Proof. This follows Lemma 4.8 whose hypotheses are verified by Lemmas 7.9, 7.10 and 7.12. □

For later reference, we provide the following fact.

Proposition 7.14. *Let $v : k \to T$ be a valuation and assume that k is with -1. The bend functor* $\mathrm{Bend}_v : \mathrm{Sch}_k \to \mathrm{Sch}_{\mathbb{T}}$ *preserves open and closed immersions.*

Proof. That Bend_v preserves open immersions follows immediately from Lemma 7.9. The question of whether Bend_v preserves closed immersions can be reduced to the affine case by choosing an open affine covering. Thus it suffices to consider a surjection $f : B \to C$ of ordered blue k-algebras and to show that $\mathrm{Bend}_v(f)$ is a surjection. But this follows at once from the definition of $\mathrm{Bend}_v(f)$ as the map $f\otimes \mathrm{id}_T : \big(B^\bullet \otimes_{k^\bullet} T /\!\!/ \mathrm{bend}_v(B)\big) \to \big(C^\bullet \otimes_{k^\bullet} T /\!\!/ \mathrm{bend}_v(C)\big)$. □

Remark 7.15. The condition that k is with -1 is not essential and can be omitted if we employ theories of ordered blue schemes that are based on alternative definitions of the spectrum of an ordered blueprint. We present two such approaches.

As explained in the earlier version [34] of this paper, the bend extends to a functor $\mathrm{Bend}_v : \mathrm{Sch}_k^{\mathrm{can}} \to \mathrm{Sch}_T^{\mathrm{can}}$ between corresponding theories of subcanonical ordered blue schemes without any restrictions on k. The reason for us not pursue with this line of thought is that the embedding $\mathrm{Rings} \to \mathrm{OBlpr}$ does not extend to an embedding of $\mathrm{Sch}_{\mathbb{Z}}^{+}$ to $\mathrm{Sch}_{\mathbb{F}_1}^{\mathrm{can}}$, and therefore subcanonical ordered blue schemes are not suitable to deal with analytifications.

The following alternative does not run into this problem, but requires more explanations. Analogously to the definition of $\mathrm{Spec}\, B$, which consists of the prime k-ideals of B, one can define the *ideal spectrum* $\mathrm{Spec}^{\mathrm{id}} B$ of B that consists of all prime ideals of B, i.e. all subsets $\mathfrak{p}$ of B such that $0 \in \mathfrak{p}$, such that $\mathfrak{p}B = \mathfrak{p}$, such that $a \equiv \sum b_j$ with $b_j \in \mathfrak{p}$ implies $a \in \mathfrak{p}$ and such that $S = B - \mathfrak{p}$ is a multiplicative set. This leads to the theory of *ordered blue ideal schemes* whose category $\mathrm{Sch}_{\mathbb{F}_1}^{\mathrm{id}}$ has similar properties to $\mathrm{Sch}_{\mathbb{F}_1}$. Some more details on such an approach can be found in the third arXiv version of [31]. In particular, the category $\mathrm{Sch}_{\mathbb{Z}}^{+}$ of usual schemes occurs as a full subcategory of $\mathrm{Sch}_{\mathbb{F}_1}^{\mathrm{id}}$; even better, the category $\mathrm{Sch}_{\mathbb{N}}^{+,\mathrm{can}}$ of subcanonical semiring schemes is a full subcategory of $\mathrm{Sch}_{\mathbb{F}_1}^{\mathrm{id}}$.

Since every k-ideal is an ideal of an ordered blueprint, the spectrum $\mathrm{Spec}\, B$ is naturally a subspace of the ideal spectrum $\mathrm{Spec}^{\mathrm{id}} B$. The restriction to this subspace induces a functor $\mathcal{F} : \mathrm{Sch}_{\mathbb{F}_1}^{\mathrm{id}} \to \mathrm{Sch}_{\mathbb{F}_1}$. In general, not every ideal is a k-ideal, but if B is an ordered blue k-algebra with -1, then this is the case, and both notions of spectra coincide. This means that $\mathcal{F}$ induces an equivalence of categories $\mathcal{F}_k : \mathrm{Sch}_k^{\mathrm{id}} \to \mathrm{Sch}_k$ if k is with -1.

Using this point of view, we can extend the functor Bend_v to valuations $v : k \to T$, without any restrictions on the ordered blueprints k and T, as a functor $\mathrm{Bend}_v : \mathrm{Sch}_k^{\mathrm{id}} \to \mathrm{Sch}_T$. The proof of Lemma 7.12 makes clear that we cannot avoid the use of two different scheme theories in such generality if we want to include the bend of usual schemes, which is desirable for analytifications. We refrain from such an approach, even though it is more conceptual, in order to keep the necessary background in scheme theory in this paper as small as possible.

7.4. Tropicalization for an idempotent tropicalization base.

We show that the bend functor is a tropicalization if the tropicalization base is idempotent.

Theorem 7.16. *Let $v : k \to T$ be a valuation in an idempotent ordered blueprint T. Let B be an ordered blue k-algebra. Then there is a canonical isomorphism*

$$\mathrm{Bend}_v(B) \quad \xrightarrow{\ \sim\ } \quad (B^{\mathrm{mon}} \otimes_{k^{\mathrm{mon}}} T^{\mathrm{pos}})^{\mathrm{core}} \otimes_{T^{\mathrm{core}}} T$$

of ordered blue T-algebras, and $\mathrm{Trop}_v(B) = \mathrm{Bend}_v(B)$ is a tropicalization of B along v.

Proof. Let $B = A /\!\!/ \mathcal{R}$ be a representation of B and $B_v = B^{\mathrm{mon}} \otimes_{k^{\mathrm{mon}}} T^{\mathrm{pos}}$. Then $\mathrm{Val}_v(B,S)$ equals the morphism set $\mathrm{Hom}_T(B_v, S^{\mathrm{pos}})$ of ordered blue T-algebras, and there are surjections

$$A \otimes_{k^\bullet} T \quad \longrightarrow \quad B_v \qquad \text{and} \qquad A \otimes_{k^\bullet} T \quad \longrightarrow \quad \mathrm{Bend}_v(B)$$

of ordered blue T-algebras. Note further that the canonical morphism $B_v^{\mathrm{core}} \to B_v$ is a bijection, which factors into bijections

$$B_v^{\mathrm{core}} \quad \longrightarrow \quad B_v^{\mathrm{core}} \otimes_{T^{\mathrm{core}}} T \quad \longrightarrow \quad B_v.$$

Every ordered blue T-algebra S is idempotent. Thus by Corollary 2.16, $S^{\mathrm{core}} \to (S^{\mathrm{pos}})^{\mathrm{core}}$ is an isomorphism and $S \to S^{\mathrm{pos}}$ is a bijection.

Putting these facts together, we see that each of the homomorphism sets

$$\mathrm{Hom}_T(B_v, S^{\mathrm{pos}}), \quad \mathrm{Hom}_T(B_v^{\mathrm{core}} \otimes_{T^{\mathrm{core}}} T, S) \quad \text{and} \quad \mathrm{Hom}_T(\mathrm{Bend}_v(B), S)$$

of ordered blue T-algebras embeds canonically into $\mathrm{Hom}_T(A\otimes_{K^\bullet}T,S^{\mathrm{pos}})$. We claim that these three homomorphism sets are equal as subsets of $\mathrm{Hom}_T(A\otimes_{K^\bullet}T,S^{\mathrm{pos}})$.

Once we have proven this, it follows that $\mathrm{Bend}_v(B)$ is isomorphic to $B_v^{\mathrm{core}}\otimes_{T^{\mathrm{core}}}T$ and represents the functor $\mathrm{Val}_v(B,-)=\mathrm{Hom}_T(B_v,-)$. Note that we can describe the isomorphism $f:\mathrm{Bend}_v(B)\to B_v^{\mathrm{core}}\otimes_{T^{\mathrm{core}}}T$ explicitly as the following map: for an element $a\in\mathrm{Bend}_v(B)$, choose an inverse image a' in $A\otimes_{k^\bullet}T$ and define $f(a)$ as the image of a' in $B_v^{\mathrm{core}}\otimes_{T^{\mathrm{core}}}T$.

We prove the equality of the three homomorphism sets in question by circular inclusions. We begin with the inclusion $\mathrm{Hom}_T(B_v,S^{\mathrm{pos}})\subset\mathrm{Hom}_T(B_v^{\mathrm{core}}\otimes_{T^{\mathrm{core}}}T,S)$. The map

$$(-)^{\mathrm{core}}:\ \mathrm{Hom}_T(B_v,S^{\mathrm{pos}})\quad\longrightarrow\quad\mathrm{Hom}_{T^{\mathrm{core}}}(B_v^{\mathrm{core}},(S^{\mathrm{pos}})^{\mathrm{core}}),$$

is injective since the canonical morphisms $B_v^{\mathrm{core}}\to B_v$ and $(S^{\mathrm{pos}})^{\mathrm{core}}\to S^{\mathrm{pos}}$ are bijections. The inclusion $(S^{\mathrm{pos}})^{\mathrm{core}}=S^{\mathrm{core}}\hookrightarrow S$ yields an inclusion

$$\mathrm{Hom}_{T^{\mathrm{core}}}(B_v^{\mathrm{core}},(S^{\mathrm{pos}})^{\mathrm{core}})\quad\subset\quad\mathrm{Hom}_{T^{\mathrm{core}}}(B_v^{\mathrm{core}},S)$$

whose codomain is equal to $\mathrm{Hom}_T(B_v^{\mathrm{core}}\otimes_{T^{\mathrm{core}}}T,S)$. Altogether this yields the desired inclusion $\mathrm{Hom}_T(B_v,S^{\mathrm{pos}})\subset\mathrm{Hom}_T(B_v^{\mathrm{core}}\otimes_{T^{\mathrm{core}}}T,S)$.

We proceed with the inclusion $\mathrm{Hom}_T(B_v^{\mathrm{core}}\otimes_{T^{\mathrm{core}}}T,S)\subset\mathrm{Hom}_T(\mathrm{Bend}_v(B),S)$. Since $\mathrm{Bend}_v(B)=A\otimes_{k^\bullet}T/\!\!/\mathrm{bend}_v(B)$, this inclusion follows if we can show that the subaddition of B_v^{core} contains the bend relation $\mathrm{bend}_v(B)$. Consider $a\leqslant\sum b_j$ in B. Then $a\otimes 1\leqslant\sum b_j\otimes 1$ in $B_v=B^{\mathrm{mon}}\otimes_{k^{\mathrm{mon}}}T^{\mathrm{pos}}$. Since T^{pos} is idempotent and totally positive, B_v is so, too. This implies that

$$a\otimes 1\,+\,\sum b_j\otimes 1\quad\leqslant\quad\sum b_j\otimes 1\,+\,\sum b_j\otimes 1\quad\equiv\quad\sum b_j\otimes 1$$

and

$$\sum b_j\otimes 1\quad\equiv\quad 0\,+\,\sum b_j\otimes 1\quad\leqslant\quad a\otimes 1\,+\,\sum b_j\otimes 1.$$

Thus $\sum b_j\otimes 1\equiv a\otimes 1+\sum b_j\otimes 1$ in B_v^{core}. This shows that the subaddition of B_v^{core} contains $\mathrm{bend}_v(B)$.

We proceed with the inclusion $\mathrm{Hom}_T(\mathrm{Bend}_v(B),S)\subset\mathrm{Hom}_T(B_v,S^{\mathrm{pos}})$. Consider a T-morphism $f:\mathrm{Bend}_v(B)\to S$. We can represent B_v as $A\otimes_{k^\bullet}T^{\mathrm{pos}}/\!\!/\mathcal{R}_v$ where $\mathcal{R}_v$ is the relation on $A\otimes_{k^\bullet}T^{\mathrm{pos}}$ that is generated by the image of the relation of B^{mon} in B_v. The composition

$$f':\ A\otimes_{k^\bullet}T\quad\longrightarrow\quad\mathrm{Bend}_v(B)\quad\longrightarrow\quad S\quad\longrightarrow\quad S^{\mathrm{pos}}$$

induces a T-morphism $B_v\to S^{\mathrm{pos}}$ if f' maps the relation $\mathcal{R}_v$ to the subaddition of S^{pos}. This can be verified on the generators of $\mathcal{R}_v$, i.e. we have to consider only the relations of the form $a\otimes 1\leqslant\sum b_j\otimes 1$ for which $a\leqslant\sum b_j$ in B. The latter relation yields the bend relation

$$a\otimes 1\,+\,\sum b_j\otimes 1\quad\equiv\quad\sum b_j\otimes 1$$

in $\mathrm{Bend}_v(B)$. Consequently, S contains the relation

$$f(a\otimes 1)\,+\,\sum f(b_j\otimes 1)\quad\equiv\quad\sum f(b_j\otimes 1),$$

and by Lemma 2.10, we have $f'(a\otimes 1)\leqslant\sum f'(b_j\otimes 1)$ in S^{pos}. This shows that f' induces a T-morphism $B_v\to S^{\mathrm{pos}}$, which implies the last inclusion and completes the proof of the theorem. □

Let $v:k\to T$ be a valuation and X an ordered blue k-scheme. We obtain the following immediate consequences.

Corollary 7.17. *If T is algebraic and idempotent, then* $\mathrm{Trop}_v(B)$ *is algebraic and isomorphic to* $\mathrm{Bend}_v(B)=(B^{\mathrm{mon}}\otimes_{k^{\mathrm{mon}}}T^{\mathrm{pos}})^{\mathrm{core}}$. □

A comparison of Theorem 7.16 with Theorem 7.4 yields:

Corollary 7.18. *If T is totally positive and idempotent, then* $\mathrm{Bend}_v(B)=B^{\mathrm{mon}}\otimes_{k^{\mathrm{mon}}}T$ *is a tropicalization of B along v.* □

Remark 7.19. It seems odd that we need two completely different proofs for the existence of $\mathrm{Trop}_v(B)$ in the case that T is totally positive and the case that T is idempotent. The question whether there is a unified proof that extends to a possibly larger class of tropical bases suggests itself. Note, however, that the non-obvious identity

$$(B^{\mathrm{mon}} \otimes_{k^{\mathrm{mon}}} T^{\mathrm{pos}})^{\mathrm{core}} \otimes_{T^{\mathrm{core}}} T \quad = \quad B^{\mathrm{mon}} \otimes_{k^{\mathrm{mon}}} T^{\mathrm{pos}}$$

for totally positive and idempotent T, which results from the two different proofs, fails to hold for a merely totally positive T or a merely idempotent T.

Provided that k is with -1, we obtain the geometric version of Theorem 7.16.

Theorem 7.20. *Let $v : k \to T$ be a valuation from an ordered blueprint k with -1 in an idempotent ordered blueprint T. Let X be an ordered blue k-scheme. Then $\mathrm{Trop}_v(X) = \mathrm{Bend}_v(X)$ is a tropicalization of X along v.*

Proof. This follows immediately from Theorem 7.4, using the construction of Bend_v in Theorem 7.13, which requires that k is with -1. Note that $Val_v(\mathrm{Spec}\, B, Y) = \mathrm{Val}_v(B, \Gamma Y)$ for any ordered blue k-algebra B if Y is affine. □

Remark 7.21. Note that the analogue of Theorem 7.20 for subcanonical ordered blue schemes does not require that k is with -1 since Bend_v due to the rigid topology of ordered blue schemes; cf. the last paragraph of section 6.1. Moreover, in this latter approach, the functor $(-)^{\mathrm{mon}}$ extends to a functor $(-)^{\mathrm{mon}} : \mathrm{Sch}_{\mathbb{F}_1} \to \mathrm{Sch}_{\mathbb{F}_1}$ and we gain a natural isomorphism $\mathrm{Bend}_v(X) \simeq (X^{\mathrm{mon}} \otimes_{k^{\mathrm{mon}}} T^{\mathrm{pos}})^{\mathrm{core}} \otimes_{T^{\mathrm{core}}} T$ for every ordered blue k-scheme X.

To point this out once more, we do not follow this line of thought in the present text since it does not cover analytifications since the bend of a usual scheme is not well-defined as a subcanonical semiring scheme.

8. Berkovich analytification

To begin with, we review the definition of the Berkovich analytification as a topological space. See [8] or [3] for more details.

Let k be a field with (non-archimedean) valuation $v : k \to \mathbb{T}$ and X a k-scheme. We consider $\mathbb{T}$ together with its real topology, cf. Example 5.3.

If $X = \mathrm{Spec}\, B$ is affine, then the *Berkovich analytification of* X is the set X^{an} of all valuations $w : B \to \mathbb{T}$ whose restriction to k is v, endowed with the compact-open topology with respect to the discrete topology on B. In other words, X^{an} is equipped with the coarsest topology such that the evaluation maps

$$\begin{array}{cccc} \mathrm{ev}_a : & X^{\mathrm{an}} & \longrightarrow & \mathbb{T} \\ & w & \longmapsto & w(a) \end{array}$$

are continuous for all $a \in B$.

If X is an arbitrary k-scheme and $\mathcal{U}$ an affine presentation of X, then we define $X^{\mathrm{an}} = \mathrm{colim}\, \mathcal{U}^{\mathrm{an}}$ as a topological space. Note that this definition is independent of the chosen affine presentation since $\mathbb{T}$ is geometrically local and with open unit group, cf. Theorem 5.2.

If we endow the set $\mathrm{Bend}_v(X)(\mathbb{T})$ of tropical points of the bend of X along v with its fine topology, then we obtain the following interpretation of the Berkovich space X^{an}.

Theorem 8.1. *The Berkovich space X^{an} is naturally homeomorphic to* $\mathrm{Bend}_v(X)(\mathbb{T})$.

Proof. In the notation of section 7, we have $X^{\mathrm{an}} = \mathrm{Val}_v(X, \mathbb{T})$ as point sets. Since $\mathbb{T}$ is idempotent, Theorem 7.16 yields that $\mathrm{Bend}_v(X)(\mathbb{T})$ is the tropicalization of X along v, i.e. $\mathrm{Bend}_v(X)(\mathbb{T}) = \mathrm{Val}_v(X, \mathbb{T})$ as point sets. This shows that X^{an} stays naturally in bijection with $\mathrm{Bend}_v(X)(\mathbb{T})$.

Since $\mathbb{T}$ is a topological Hausdorff semifield with respect to the real topology, cf. Example 5.3, the fine topology of $\mathrm{Bend}_v(X)(\mathbb{T})$ is determined by any affine open covering of X. By

virtue of Theorem 5.2, this reduces the comparison of the topologies of X^{an} and $\mathrm{Bend}_v(X)(\mathbb{T})$ to the case of an affine k-scheme $X = \operatorname{Spec} B$.

In this case, the canonical bijection $\Psi : \mathrm{Bend}_v(B)(\mathbb{T}) \to B^{\mathrm{an}}$ is given explicitly as

$$\Big(f : \mathrm{Bend}_v(B) \longrightarrow \mathbb{T}\Big) \quad \longmapsto \quad \Big(w : B \xrightarrow{1:1} B^{\bullet} \longrightarrow \mathrm{Bend}_v(B) \xrightarrow{f} \mathbb{T}\Big)$$

where we use that $\mathrm{Bend}_v(B)$ is defined as $B^{\bullet} \otimes_{k^{\bullet}} \mathbb{T} /\!\!/ \mathrm{bend}_v(B)$. By definition of the affine topology, the topology of $\mathrm{Bend}_v(X)(\mathbb{T})$ is generated by subsets of the form

$$U_{a\otimes t, W} \quad = \quad \big\{ f : \mathrm{Bend}_v(B) \to \mathbb{T} \,\big|\, f(a\otimes t) \in W \big\}$$

where $a \in B$, $t \in \mathbb{T}$ and $W \subset \mathbb{T}$ is an open subset, while the topology of B^{an} is generated by subsets of the form

$$U_{a,W} \quad = \quad \big\{ w : B \to \mathbb{T} \,\big|\, w(a) \in W \big\}$$

where $a \in B$ and $W \subset \mathbb{T}$ is an open subset. It is easily verified that $\Psi^{-1}(U_{a,W}) = U_{a\otimes 1,W}$. Thus Ψ is continuous.

We are left with proving that Ψ is open. Consider a basic open $U_{a\otimes t,Z}$ of $\mathrm{Bend}_v(B)(\mathbb{T})$ and denote by $m_t : \mathbb{T} \to \mathbb{T}$ the multiplication with t. Then we have

$$\begin{aligned} U_{a\otimes t,Z} \quad &= \quad \big\{ f : \mathrm{Bend}_v(B) \to \mathbb{T} \,\big|\, f(a\otimes t) \in Z \big\} \\ &= \quad \big\{ f : \mathrm{Bend}_v(B) \to \mathbb{T} \,\big|\, f(a\otimes 1) \in m_t^{-1}(Z) \big\} \quad = \quad U_{a\otimes 1, m_t^{-1}(Z)}. \end{aligned}$$

Since the multiplication of $\mathbb{T}$ is continuous, $m_t^{-1}(Z)$ is an open subset of T. Thus we see that $\Psi(U_{a\otimes t,Z}) = U_{a,m_t^{-1}(Z)}$ is open in B^{an}, which concludes the proof of the theorem. □

Remark 8.2. The same technique of proof can be used to show that X^{an} is homeomorphic to $\mathrm{Hom}_{\mathbb{T}}(X^{\mathrm{mon}} \otimes_{k^{\mathrm{mon}}} \mathbb{T}^{\mathrm{pos}}, \mathbb{T}^{\mathrm{pos}})$. This viewpoint can be extended to the Berkovich space $\mathcal{M}(B)$ of all, possibly archimedean, valuations of a ring B as follows. Also confer the work Berkovich ([8, Section 1]) and Poineau ([46]) on the Berkovich space $\mathcal{M}(\mathbb{Z})$ of the arithmetic line.

Consider the trivial valuation $v : \mathbb{F}_1 \to \mathbb{F}_1$. Then every valuation $w : B \to \mathbb{R}_{\geqslant 0}$ is an extension of v. Define $B_v = B^{\mathrm{mon}} \otimes_{\mathbb{F}_1^{\mathrm{mon}}} \mathbb{F}_1^{\mathrm{pos}} = (B^{\mathrm{mon}})^{\mathrm{pos}}$ and consider $\mathbb{R}_{\geqslant 0}^{\mathrm{pos}}$ with the real topology. Then $\mathcal{M}(B)$ is naturally homeomorphic to $\mathrm{Hom}_{\mathbb{F}_1}(B_v, \mathbb{R}_{\geqslant 0}^{\mathrm{pos}})$, endowed with the fine topology.

9. Kajiwara-Payne tropicalization

Before we explain how to recover the Kajiwara-Payne tropicalization as a rational point set of a scheme theoretic tropicalization, we review the definition of toric varieties and the theory of Kajiwara ([26]) and Payne ([45]). This shall serve the reader as a reminder and allows us to fix notation.

9.1. Toric varieties. Let $N_{\mathbb{R}}$ be a real vector space with a lattice $N_{\mathbb{Z}}$. A *(strongly convex polyhedral rational) cone in* $N_{\mathbb{R}}$ is a convex cone τ of $N_{\mathbb{R}}$ such that $\tau \cap (-\tau) = \{0\}$ and that is generated over $\mathbb{R}_{\geqslant 0}$ by finitely many elements of $N_{\mathbb{Z}}$. A *face of* τ is the intersection of τ with a half space H of $\mathbb{N}_{\mathbb{R}}$ such that the intersection is either equal to τ or is contained in the boundary of τ. A *fan in* $N_{\mathbb{R}}$ is a collection Δ of cones in $N_{\mathbb{R}}$ such that every face of a cone in Δ is in Δ and such that the intersection of two cones in Δ is a face of each of these cones.

The *dual cone* of a cone τ is the subsemigroup $\tau^{\vee} = \{x \in N_{\mathbb{R}}^{\vee} | <x,y> \geqslant 0 \text{ for all } y \in \tau\}$ of the dual vector space $N_{\mathbb{R}}^{\vee}$. We define the monoid A_τ with 0 that results from writing $\tau^{\vee} \cap N_{\mathbb{Z}}^{\vee}$ multiplicatively and adjoining an additional element 0. Then an inclusion $\sigma \subset \tau$ of cones in Δ yields a finite localization $A_\tau \to A_\sigma$ of monoids. This defines a diagram $\mathcal{D}$ of monoids A_τ with zero and finite localizations.

Applying the functor Spec to the diagram $\mathcal{D}$ yields an affine presentation $\mathcal{U}$ in affine monoid schemes. The base extension $\mathcal{U}_k^+$ is an affine presentation in affine k-schemes. We define the *toric variety associated with* Δ as $X(\Delta) = \operatorname{colim} \mathcal{U}_k^+$.

9.2. Tropicalization of closed subvarieties. We begin with the tropicalization of an affine toric variety. Let τ be a cone in $N_\mathbb{R}$ and $U_\tau = \operatorname{Spec} A_\tau$. Then $X_\tau = U^+_{\tau,k}$ is an affine toric variety. Its analytification X_τ^{an} is the set $\mathrm{Val}_v(k[A_\tau]^+,\mathbb{T})$ of all valuations $w : k[A_\tau]^+ \to \mathbb{T}$ that extend v, endowed with the compact-open topology. The *Kajiwara-Payne tropicalization* $\mathrm{Trop}_v^{KP}(X_\tau)$ *of* X_τ is defined as the set $\mathrm{Hom}(A_\tau,\mathbb{T})$ of monoid morphisms, endowed with the compact-open topology where we regard A_τ as a discrete monoid. It comes with the continuous surjection

$$\mathrm{trop}_{v,\tau}^{KP}: \quad X_\tau^{\mathrm{an}} \; = \; \mathrm{Val}_v(k[A_\tau]^+,\mathbb{T}) \quad \longrightarrow \quad \mathrm{Hom}(A_\tau,\mathbb{T}) \; = \; \mathrm{Trop}_v^{KP}(X_\tau)$$

that restricts a valuation $w : k[A_\tau]^+ \to \mathbb{T}$ to the monoid morphism $w|_{A_\tau} : A_\tau \to \mathbb{T}$.

This construction is compatible with glueing affine pieces along principal open subsets. Namely, let Δ be a fan in $N_\mathbb{R}$. Then $X(\Delta) = \operatorname{colim} X_\tau$, with respect to the inclusions $X_\sigma \subset X_\tau$ whenever $\sigma \subset \tau$, and $X(\Delta)^{\mathrm{an}} = \operatorname{colim} X_\tau^{\mathrm{an}}$ as topological space. We define $\mathrm{Trop}_v^{KP}(X(\Delta))$ as colimit of the topological spaces $\mathrm{Hom}(A_\tau,\mathbb{T})$ with respect to the open topological embeddings $\mathrm{Hom}(A_\sigma,\mathbb{T}) \to \mathrm{Hom}(A_\tau,\mathbb{T})$ that are induced by inclusions $\sigma \subset \tau$. Moreover the map $X_\tau^{\mathrm{an}} \to \mathrm{Trop}_v^{KP}(X_\tau)$ extends to a continuous surjection

$$\mathrm{trop}_{v,\Delta}^{KP}: \quad X(\Delta)^{\mathrm{an}} \quad \longrightarrow \quad \mathrm{Trop}_v^{KP}(X(\Delta)).$$

A closed immersion $\iota : Y \to X(\Delta)$ of a k-scheme Y into the toric k-variety $X(\Delta)$ yields a closed embedding $\iota^{\mathrm{an}} : Y^{\mathrm{an}} \to X(\Delta)^{\mathrm{an}}$ of topological spaces. We define the *Kajiwara-Payne tropicalization* $\mathrm{Trop}_{v,\iota}^{KP}(Y)$ *of* Y as the image $\mathrm{trop}_{v,\Delta}^{KP}(Y)$, endowed with the subspace topology with respect to the inclusion $\iota^{\mathrm{trop}} : \mathrm{Trop}_{v,\iota}^{KP}(Y) \to \mathrm{Trop}_v^{KP}(X(\Delta))$. The Kajiwara-Payne tropicalization of Y comes with a surjective continuous map

$$\mathrm{trop}_{v,\iota}^{KP}: \quad Y^{\mathrm{an}} \quad \longrightarrow \quad \mathrm{Trop}_{v,\iota}^{KP}(Y).$$

9.3. The associated blue scheme. We explain how to obtain $\mathrm{Trop}_{v,\iota}^{KP}(Y)$ as a set of rational points of a scheme theoretic tropicalization $\mathrm{Trop}_v(Z)$ of a suitable blue k-scheme Z that we define in the following.

If $X(\Delta) = X_\tau = \operatorname{Spec} k[A_\tau]^+$ is affine, then the closed immersion $\iota : Y \to X_\tau$ corresponds to a surjection $\pi : k[A_\tau]^+ \to \Gamma Y$ of rings. We define the blue k-scheme Z_τ as $\operatorname{Spec} k[A_\tau] /\!\!/ \mathcal{R}_\tau$ where

$$\mathcal{R}_\tau \quad = \quad \Big\{ \sum a_i \equiv \sum b_j \;\Big|\; \sum \pi(a_i) = \sum \pi(b_j) \Big\}.$$

Note that the natural inclusion $k[A_\tau] /\!\!/ \mathcal{R}_\tau \to \Gamma Y$ induces a morphism $\beta_\tau : Y \to Z_\tau$ whose associated morphism $\beta_\tau^+ : Y \to Z_\tau^+$ of k-schemes is an isomorphism.

For a closed subscheme Y of an arbitrary toric k-variety $X(\Delta)$, we define the affine presentation $\mathcal{V}_k(\Delta)$ as the diagram of morphisms $Z_\sigma \to Z_\tau$ whenever $\sigma \subset \tau$ and we define the blue k-scheme Z as $\operatorname{colim} \mathcal{V}_k(\Delta)$. The morphisms β_τ glue to a morphism $\beta : Y \to Z$, which induces an isomorphism $\beta^+ : Y \to Z^+$ of k-schemes. We say that Z is the *blue model of* Y *induced by* ι.

Theorem 9.1. *The Kajiwara-Payne tropicalization* $\mathrm{Trop}_{v,\iota}^{KP}(Y)$ *is naturally homeomorphic to* $\mathrm{Bend}_v(Z)(\mathbb{T})$ *and the diagram*

$$\begin{tikzcd} Y^{\mathrm{an}} \arrow[r, "\mathrm{trop}_{v,\Delta}^{KP}"] \arrow[d, "\simeq"] & \mathrm{Trop}_{v,\iota}^{KP}(Y) \arrow[d, "\simeq"] \\ \mathrm{Bend}_v(Y)(\mathbb{T}) \arrow[r, "\mathrm{Bend}_v(\beta)(\mathbb{T})"] & \mathrm{Bend}_v(Z)(\mathbb{T}) \end{tikzcd}$$

of continuous maps commutes.

Proof. Since all functors in questions are defined in terms of affine coverings, it is enough to prove the theorem in the affine case. Since the set $X_\tau^{\mathrm{trop}} = \mathrm{Hom}(k[A_\tau],\mathbb{T})$ of monoid morphisms stays in natural bijection with the set $\mathrm{Val}_v(k[A_\tau],\mathbb{T})$ of valuations on $k[A_\tau]$ in $\mathbb{T}$ that extend v,

the image $\mathrm{Trop}^{KP}_{v,\iota}(Y)$ of $Y^{\mathrm{an}} = \mathrm{Val}_v(k[A_\tau]/I,\mathbb{T})$ under $\mathrm{Trop}^{KP}_{v,\tau}$ equals $\mathrm{Val}_v(k[A_\tau]/\!\!/\mathcal{R}_v,\mathbb{T})$, which stays in natural bijection with $\mathrm{Bend}_v(Z)(\mathbb{T})$ by Theorem 7.16.

Note that the commutativity of the diagram follows from the functoriality of the bend functor. Thus we are left with showing that the bijection $\mathrm{Trop}^{KP}_{v,\iota}(Y) \to \mathrm{Bend}_v(Z)(\mathbb{T})$ is a homeomorphism. By Theorem 11.2 and Example 5.3 and the local definition of the topology of $\mathrm{Trop}^{KP}_{v,\iota}(Y)$, this can be verified on affine patches.

Since $\mathrm{Bend}_v(Z)(\mathbb{T})$ inherits the subspace topology from $\mathrm{Hom}_{\mathbb{T}}(\mathrm{Bend}_v(k[A_\tau]),\mathbb{T})$ and $\mathrm{Trop}^{KP}_{v,\iota}(Y)$ is defined as a topological subspace of $X^{\mathrm{trop}}_\tau = \mathrm{Hom}(A_\tau,\mathbb{T})$, it suffices to show that the natural bijection

$$\Psi:\ \mathrm{Hom}(A_\tau,\mathbb{T}) \quad\longrightarrow\quad \mathrm{Hom}_{\mathbb{T}}(\mathrm{Bend}_v(k[A_\tau]),\mathbb{T})$$

is a homeomorphism. Since $\mathrm{Bend}_v(k[A_\tau]) = k[A_\tau]^\bullet \otimes_{k^\bullet} \mathbb{T}/\!\!/\mathrm{bend}_v(k[A_\tau])$, the image of a basic open of the former morphism set is

$$\Psi(U_{a,V}) \quad=\quad \big\{\, f : k[A_\tau]^\bullet \otimes_{k^\bullet} \mathbb{T} \to \mathbb{T} \,\big|\, f(a\otimes 1) \in V \,\big\} \quad=\quad U_{a\otimes 1,V}$$

where $a \in A_\tau$ and $V \subset \mathbb{T}$ open, and $U_{a\otimes 1,V}$ is a basic open in $\mathrm{Hom}_{\mathbb{T}}(\mathrm{Bend}_v(k[A_\tau]),\mathbb{T})$. This shows that Ψ is an open map.

Conversely, consider a basic open $U_{ca\otimes t,V}$ of $\mathrm{Hom}_{\mathbb{T}}(\mathrm{Bend}_v(k[A_\tau]),\mathbb{T})$ where $c \in k$, $a \in A_\tau$, $t \in \mathbb{T}$ and $V \subset \mathbb{T}$ is an open subset. Denote by $m_{v(c)t} : \mathbb{T} \to \mathbb{T}$ the multiplication by $v(c)t$, which is a continuous map. Then we have

$$\begin{aligned} U_{ca\otimes t,V} \quad &=\quad \big\{\, f : \mathrm{Bend}_v(k[A_\tau]) \to \mathbb{T} \,\big|\, f(ca\otimes t) \in V \,\big\} \\ &=\quad \big\{\, f : \mathrm{Bend}_v(k[A_\tau]) \to \mathbb{T} \,\big|\, f(a\otimes 1) \in m^{-1}_{v(c)t}(V) \,\big\} \quad=\quad U_{a\otimes 1, m^{-1}_{v(c)t}(V)} \end{aligned}$$

and thus $\Psi^{-1}(U_{ca\otimes t,V}) = U_{a,m^{-1}_{v(c)t}(V)}$, which is a basic open of $\mathrm{Hom}(A_\tau,\mathbb{T})$. This shows that Ψ is continuous and completes the proof of the theorem. □

10. Foster-Ranganathan tropicalization

Motivated by the paper [6] of Banerjee, Foster and Ranganathan generalize in [19] the Kajiwara-Payne tropicalization to higher rank valuations of the ground field k. In this section, we will show how the Foster-Ranganathan tropicalization fits into the context of scheme theoretic tropicalization. We recall the setup of [19].

We denote by $\mathbb{T}^{(n)}$ the idempotent semiring $\mathbb{R}^n_{>0} \cup \{0\}$ with the componentwise multiplication, whose addition addition is defined as taking the maximum with respect to the lexicographical order. The order topology turns $\mathbb{T}^{(n)}$ into a topological Hausdorff semifield. In particular, $\mathbb{T}^{(n)}$ satisfies all the hypotheses of Theorem 5.2.

Let k be a field with valuation $v : k \to \mathbb{T}^{(n)}$ and $Y = \mathrm{Spec}\, B$ an affine k-scheme. The *Foster-Ranganathan analytification of Y along v* is the set

$$\mathrm{An}^{FR}_v(Y) \;=\; \{\, w : B \to \mathbb{T}^{(n)} \,|\, w|_k = v_k \,\}$$

of all valuations w that extend v to B. It is endowed with the compact-open topology where B is considered as a discrete k-algebra.

Let A be a commutative monoid with zero and $\pi : k[A]^+ \to B$ a surjection of k-algebras. This corresponds to a closed embedding $\iota : Y \to X$ of k-schemes where $X = \mathrm{Spec}\, k[A]^+$. For instance, if $A - \{0\}$ is a finitely generated abelian group, then X is a split k-torus. The *Foster-Ranganathan tropicalization of Y along v with respect to ι* is the image $\mathrm{Trop}^{FR}_{v,\iota}(Y)$ of the map

$$\mathrm{trop}^{FR}_{v,\iota} : \mathrm{An}^{FR}_v(Y) \;\longrightarrow\; \mathrm{Hom}(A,\mathbb{T}^{(n)})$$

that sends a valuation $w : B \to \mathbb{T}^{(n)}$ to the composition $A \to k[A]^+ \to B \to \mathbb{T}^{(n)}$. We endow $\mathrm{Hom}(A,\mathbb{T}^{(n)})$ with the compact-open topology, with respect to the discrete topology for A, and $\mathrm{Trop}^{FR}_{v,\iota}(Y)$ with the subspace topology.

Let $Z = \mathrm{Spec}\, k[A]/\!\!/\mathcal{R}$ be the blue k-scheme associated with $\iota : Y \to X$ where $\mathcal{R} = \{\sum a_i \equiv \sum b_j | \sum \pi(a_i) = \sum \pi(B_j)\}$, and $\beta : Y \to Z$ the morphism induced by the inclusion $k[A]/\!\!/\mathcal{R} \to B$.

Theorem 10.1. *The Foster-Ranganathan analytification* $\mathrm{An}^{FR}_v(Y)$ *is naturally homeomorphic to* $\mathrm{Bend}_v(Y)(\mathbb{T}^{(n)})$*, the Foster-Ranganathan tropicalization* $\mathrm{Trop}^{FR}_{v,\iota}(Y)$ *is naturally homeomorphic to* $\mathrm{Bend}_v(Z)(\mathbb{T}^{(n)})$ *and the diagram*

$$\begin{tikzcd} \mathrm{An}^{FR}_v(Y) \arrow[r, "\mathrm{trop}^{FR}_{v,\iota}"] \arrow[d, "\simeq"] & \mathrm{Trop}^{FR}_{v,\iota}(Y) \arrow[d, "\simeq"] \\ \mathrm{Bend}_v(Y)(\mathbb{T}^{(n)}) \arrow[r, "\mathrm{Bend}_v(\beta)(\mathbb{T}^{(n)})"] & \mathrm{Bend}_v(Z)(\mathbb{T}^{(n)}) \end{tikzcd}$$

of continuous maps commutes.

Proof. The proofs of Theorems 8.1 and 9.1 apply verbatim with $\mathbb{T}$ exchanged by $\mathbb{T}^{(n)}$. □

Remark 10.2. Foster and Ranganathan consider in [19] also the topology for $\mathbb{T}^{(n)}$ induced from the natural embedding into the Euclidean space $\mathbb{R}^n$. Since $\mathbb{T}^{(n)}$ is also a topological Hausdorff semiring with respect to the Euclidean topology, the analogue of Theorem 10.1 also holds for the Euclidean topology.

Remark 10.3. Note that the scheme theoretic Foster-Ranganathan analytification is compatible with the order preserving projections $\mathbb{T}^{(n)} \to \mathbb{T}^{(l)}$ for $l < n$, as considered in section 2.1 of [19], in the following sense.

A rank n valuation $v_n : k \to \mathbb{T}^{(n)}$ of the field k induces a rank l valuation $v_j : k \to \mathbb{T}^{(l)}$ for every $l < n$ by composing with the projection $\pi_{n,l} : \mathbb{T}^{(n)} \to \mathbb{T}^{(l)}$ onto the first l factors. This map induces a continuous map $\pi^{FR}_{n,l}(Y) : \mathrm{An}^{FR}_{v_n}(Y) \to \mathrm{An}^{FR}_{v_l}(Y)$ of Foster-Ranganathan analytifications.

This continuous map can be recovered by applying $\Phi_Y = \mathrm{Hom}_{\mathbb{T}^{(n)}}(\mathrm{Spec}(-), \mathrm{Bend}_{v_n}(Y))$ to the morphism $\pi_{n,l} : \mathbb{T}^{(n)} \to \mathbb{T}^{(l)}$. In other words,

$$\begin{tikzcd} \mathrm{An}^{FR}_{v_n}(Y) \arrow[r, "\pi^{FR}_{n,l}(Y)"] \arrow[d, "\sim"] & \mathrm{An}^{FR}_{v_l}(Y) \arrow[d, "\sim"] \\ \mathrm{Bend}_{v_n}(Y)(\mathbb{T}^{(n)}) \arrow[r, "\Phi_Y(\pi_{n,l})"] & \mathrm{Bend}_{v_l}(Y)(\mathbb{T}^{(l)}) \end{tikzcd}$$

is a commutative diagram of continuous maps where we use that $\mathrm{Bend}_{v_n}(Y) \otimes_{\mathbb{T}^{(n)}} \mathbb{T}^{(l)}$ is isomorphic to $\mathrm{Bend}_{v_l}(Y)$.

There is a similar commutative diagram for the projections $\mathrm{Trop}^{FR}_{v_n,\iota}(Y) \to \mathrm{Trop}^{FR}_{v_l,\iota}(Y)$ of Foster-Ranganathan tropicalizations, which play a prominent role in a forthcoming paper by Foster and Hully.

10.1. What is new? As a consequence, we can extend the Foster-Ranganathan tropicalization beyond the affine case. For instance, a closed immersion $\iota : Y \to X(\Delta)$ of Y into a toric variety $X(\Delta)$ yields an associated blue k-model Z of Y. We define the corresponding *Foster-Ranganathan tropicalization* $\mathrm{Trop}^{FR}_{v,\iota}(Y)$ as the topological space $\mathrm{Bend}_v(Z)(\mathbb{T}^{(n)})$. By Theorem 5.2, the affine open subschemes $Y_\sigma = Y \cap X_\sigma$ for $\sigma \in \Delta$ yield open topological embeddings $\mathrm{Trop}^{FR}_{v,\iota}(Y_\sigma) \to \mathrm{Trop}^{FR}_{v,\iota}(Y)$, which cover $\mathrm{Trop}^{FR}_{v,\iota}(Y)$.

More generally, the Foster-Ranganathan tropicalization can be extended to toroidal embeddings and log-structures via their associated blue schemes, as considered in section 15.

11. Giansiracusa tropicalization

Jeff and Noah Giansiracusa introduce in [21] the bend relation for a closed subscheme of a toric variety or, more general, of a monoidal scheme over a non-archimedean field k. We recall this theory and explain how to recover the Giansiracusa bend relation from our point of view.

11.1. The bend functor for morphisms into monoid schemes. Let $k \to R$ be a ring homomorphism and $v : k \to T$ a valuation into an idempotent semiring T that is *totally ordered*, i.e. for all $a, b \in T^{\text{pos}}$, either $a \leqslant b$ or $b \leqslant a$. Let A_0 be a monoid with zero and $\eta : A_0 \to R$ a multiplicative map such that the induced homomorphism $\eta_k^+ : k[A_0]^+ \to R$ of k-algebras is surjective, with kernel $I = (\eta_k^+)^{-1}(0)$. Then the *Giansiracusa tropicalization of R with respect to v and η* is the semiring

$$\mathrm{Trop}^{GG}_{v,\eta}(R) \quad = \quad T[A_0]^+ \,/\!\!/\, \mathrm{bend}^{GG}_{v,\eta}(I)$$

where the *Giansiracusa bend relation* $\mathrm{bend}^{GG}_{v,\eta}(I)$ is generated by the relations

$$v(c_a)a \,+\, \sum v(c_j)b_j \quad \equiv \quad \sum v(c_j)b_j \qquad \text{for which} \qquad c_a a \,+\, \sum c_j b_j \,\in\, I$$

with $c_a, c_j \in k$ and $a, b_j \in A_0$.

Remark 11.1. Note that the congruence $\mathrm{bend}^{GG}_{v,\eta}(I)$ is generated by those relations of the above form for which a and the b_j are pairwise different. This observation explains that our definition of $\mathrm{Trop}^{GG}_{v,\eta}(R)$ coincides with the original definition in [21].

For integral monoids A, [21] shows that the bend relations are compatible with localizations. Therefore the Giansiracusa tropicalization can be extended to k-schemes Y with respect to a morphism $\iota : Y \to X$ to an integral monoid scheme X that induces a closed immersion $\iota_k^+ : Y \to X_k^+$. Choosing compatible affine presentations of Y and X and applying $\mathrm{Trop}^{GG}_{v,\eta}$, where η stays for the map between the coordinate blueprints of objects of the chosen affine presentations, yields the Giansiracusa tropicalization $\mathrm{Trop}^{GG}_{v,\iota}(Y)$ of Y with respect to v and ι.

11.2. The associated blue scheme. The connection with the bend functor from section 7.3 is as follows. The map $\eta : A_0 \to k[A_0]^+ \to R$ yields the blueprint $B = A /\!\!/ \mathcal{R}$ with $A = k[A_0]$ and

$$\mathcal{R} \quad = \quad \big\{ \sum a_i \equiv \sum b_j \,\big|\, \sum \eta^+(a_i) = \sum \eta^+(b_j) \text{ in } R \,\big\},$$

together with the morphism $B \to R$, which induces an isomorphism $B^+ \simeq R$. A morphism $\iota : Y \to X$ into a monoid scheme yields a blue k-scheme Z by choosing compatible affine presentations for Y and X and applying the above definition to the coordinate blueprints.

Theorem 11.2. *There is a canonical morphism* $\mathrm{Bend}_v(Z) \to \mathrm{Trop}^{GG}_{v,\iota}(Y)$ *that induces an isomorphism* $\mathrm{Bend}_v(Z)^+ \simeq \mathrm{Trop}^{GG}_{v,\iota}(Y)$ *of semiring schemes.*

Proof. Since the definition of $\mathrm{Trop}^{GG}_{v,\iota}(Y)$ in terms of affine presentations is compatible with the definition of $\mathrm{Bend}_v(Z)$, we are reduced to the affine case of a multiplicative morphism $\eta : A_0 \to R$ from a monoid A_0 with zero into a k-algebra R that yields a surjection $\eta_k^+ : k[A_0]^+ \to R$. Let B be the associated blueprint. We have to show that there is a canonical morphism $\mathrm{Bend}_v(B) \to \mathrm{Trop}^{GG}_{v,\eta}(R)$ that induces an isomorphism $\mathrm{Bend}_v(B)^+ \simeq \mathrm{Trop}^{GG}_{v,\eta}(R)$ of semirings.

The association $c \cdot a \otimes t \mapsto v(c)t \cdot a$ for $c \in k$, $a \in A_0$ and $t \in T$ defines an isomorphism $\varphi : k[A_0]^\bullet \otimes_{k^\bullet} T \to T[A_0]$ of ordered blue T-algebras. If we can show that φ identifies the relation $\mathrm{bend}_v(B)$ on $(k[A_0]^\bullet \otimes_{k^\bullet} T)^+$ with the Giansiracusa bend relation $\mathrm{bend}^{GG}_{v,\eta}(I)$ on $T[A_0]^+$, then it follows that φ induces an isomorphism $\mathrm{Bend}_v(B) \to T[A_0] /\!\!/ \mathrm{bend}^{GG}_{v,\eta}(I)$, and the lemma follows after applying $(-)^+$.

For the identification between the bend relations, we need the fact that $v(c) = v(-c)$ for all $c \in k$. Since T is totally ordered, we have $v(c) \leqslant v(-c)$ or $v(-c) \leqslant v(c)$ in T^{pos}. However, these two relations are equivalent, as multiplication with $v(-1)$ shows:

$$v(-c) \quad = \quad v(-1)v(c) \quad \leqslant \quad v(-1)v(-c) \quad = \quad v(c)$$

if $v(c) \leqslant v(-c)$, and vice versa. This shows that $v(c) = v(-c)$ in T^{pos}. By Corollary 2.16, the canonical map $T \to T^{\mathrm{pos}}$ is a bijection. Thus $v(c) = v(-c)$ in T.

We show that $c_a a \otimes 1 + \sum c_j b_j \otimes 1 \equiv \sum c_j b_j \otimes 1$ is a generator of $\mathrm{bend}_v(B)$ if and only if $v(c_a)a + \sum v(c_j)b_j \equiv \sum v(c_j)b_j$ is a generator of $\mathrm{bend}^{GG}_{v,\eta}(I)$. Indeed, the former relation is in $\mathrm{bend}_v(B)$ if and only if $c_a a \leqslant \sum c_j b_j$ in B. Since B is algebraic and with -1, this is equivalent to $c_a a + \sum(-c_j)b_j \equiv 0$, which, in turn, is equivalent to $c_a a + \sum(-c_j)b_j \in I$. Since $v(-c) = v(c)$ for $c \in k$, the latter condition is equivalent to $v(c_a)a + \sum v(c_j)b_j \equiv \sum v(c_j)b_j$ in $\mathrm{Bend}^{GG}_{v,\eta}(R)$, as claimed. □

By Theorem 11.2, we obtain Theorem 3.3.6 of [22] as a special case of Theorem 7.16. Namely, let Alg^+_T be the category of T-algebras, which are homomorphisms $T \to S$ of semirings, and let $\mathrm{Val}^+_v(Y,-) : \mathrm{Alg}^+_T \to \mathrm{Sets}$ be the restriction of $\mathrm{Val}_v(Y,-)$ to Alg^+_T.

Corollary 11.3. *Let $v : k \to T$ be a valuation from a ring k into a totally ordered idempotent semiring T and $\iota : Y \to X$ be a morphism of a k-scheme Y into a monoid scheme X such that $\iota^+_k : Y \to X^+_k$ is a closed immersion. Then $\mathrm{Val}^+_v(Y,-)$ is represented by the semiring T-scheme $\mathrm{Trop}^{GG}_{v,\iota}(Y)$.*

Proof. Let Z be the associated blue k-scheme. By Theorem 7.16, we have $\mathrm{Val}^+_v(Z,-) = \mathrm{Hom}_T(\mathrm{Spec}(-), \mathrm{Bend}_v(Z))$. Since every morphism $\mathrm{Spec}\, S \to \mathrm{Bend}_v(Z)$ factors uniquely through a T-morphism $\mathrm{Spec}\, S \to \mathrm{Bend}_v(Z)^+$, the functor $\mathrm{Val}^+_v(Z^+,-)$ is isomorphic to the functor $\mathrm{Hom}_T(\mathrm{Spec}(-), \mathrm{Bend}_v(Z)^+)$. Therefore the corollary follows from $Y = Z^+$ and Theorem 11.2. □

11.3. What is new? To speak in an analogy, the difference between the Giansiracusa tropicalization and the scheme theoretic tropicalization in terms of blue schemes is similar to the difference between subvarieties of a projective space and projective varieties. In other words, the enhancement of a variety Y in an ambient monoid scheme with the structure of a blue scheme allows us to detach the Y from the monoid scheme and tropicalize it as an independent abstract geometric object. Besides this conceptual novelty, we have eliminated the following two technical restrictions of [21].

Theorem 11.2 guarantees that the bend functor from this paper incorporates the Giansiracusa tropicalization completely. Therefore we can extend the Giansiracusa tropicalization to morphism $\iota : Y \to X$ into monoid schemes that are not necessarily integral. This can also be seen by directly generalizing [21].

Moreover, we can generalize the Giansiracusa tropicalization to valuations $v : k \to T$ into any idempotent semiring. Note that for this generalization, it is important to adapt the sign convention for the bend relation of this text since $v(c) = v(-c)$ does not hold true for all valuations into idempotent semirings.

12. Maclagan-Rincón weights

Maclagan and Rincón show in [36] that the weights of the tropicalization of a closed subvariety of a torus can be recovered from the Giansiracusa tropicalization. We will extend the argument of [36] to the scheme theoretic tropicalizations considered in this paper.

12.1. Weights from the classical variety. Let k be a field with valuation $v : k \to \mathbb{T}$ and value group $\Gamma = v(k^\times)$. Assume that Γ is dense in $\mathbb{T}$ and that there exists a section $s : \Gamma \to k^\times$ to $v : k^\times \to \Gamma$ as a group homomorphism. Note that such a section always exists after passing to a suitable finite field extension of k.

In this situation, the tropicalization $\mathrm{Trop}(Y) = \mathrm{Trop}^{KP}_{v,\iota}(Y)$ of a closed k-subscheme Y of a split torus $\mathbb{G}^{n,+}_{m,k}$ can be endowed with the structure of a polyhedral complex whose top dimensional cells come with weights that satisfy a certain balancing condition with respect to the embedding $\mathrm{Trop}(Y) \subset (\mathbb{T}^*)^n \simeq \mathbb{R}^n$ where we identify $(\mathbb{T}^\times)^n$ with $\mathbb{R}^n$ by taking coordinatewise logarithms. Note our specific usage of $\mathbb{G}_{m,k} = \mathrm{Spec}\, k[X^{\pm 1}]$ and $\mathbb{G}^+_{m,k} = \mathrm{Spec}\, k[X^{\pm 1}]^+$, cf. section 1. In the following, we recall the definition of the weights of $\mathrm{Trop}(Y)$.

Let $\mathcal{O}_k = \{a \in k | v(a) \leqslant 1\}$ be the *integers of k*, $\mathfrak{m} = \{a \in \mathcal{O}_k | v(a) < 1\}$ its unique maximal ideal and $k_0 = \mathcal{O}_k/\mathfrak{m}$ the *residue field*. Let $\iota : Y \to \mathbb{G}^{n,+}_{m,k}$ be a closed immersion of k-schemes and $I \subset R$ the defining ideal where $R^+_n = k[X_1^{\pm 1}, \dots, X_n^{\pm 1}]^+$ is the coordinate ring of $\mathbb{G}^{n,+}_{m,k}$. Let $w = (w_i)$ be a point of the tropicalization $\mathrm{Trop}(Y) \subset \mathbb{T}^n$. For an element $f = \sum c_e X^e$ of I where $e \in \mathbb{Z}^n$ is a multi-index and $c_e \in k$, we define its *tropicalization* as $v(f) = \sum v(c_e) X^e$ and its *value at w* as $v(f)(w) = \sum v(c_e) w^e$. We denote by $\overline{a} \in k_0$ the residue class of an element $a \in \mathcal{O}_k$. The *initial form of f in w* is

$$\mathrm{in}_w(f) \quad = \sum_{v(c_e)w^e = v(f)(w)} \overline{s\big(v(c_e)^{-1}\big)\; v(c_e)}\; X^e,$$

which is an element of $\overline{R}^+_n = k_0[X_1^{\pm 1}, \dots, X_n^{\pm 1}]^+$. The *initial ideal of I in w* is the ideal $\mathrm{in}_w(I)$ of $\overline{R}^+_n$ that is generated by the initial forms $\mathrm{in}_w(f)$ of all $f \in I$. Let $\mathrm{in}_w(T) = \bigcap \mathfrak{q}_i$ be a primary decomposition for I and $\mathfrak{p}_i$ the radical of $\mathfrak{q}_i$. We denote by $\mathrm{mult}(\mathfrak{p}_i, \mathrm{in}_w(I))$ the length of the $\overline{R}^+_n$-module $(\overline{R}^+_n/\mathfrak{q}_i)_{\mathfrak{p}_i}$. The *multiplicity of* $\mathrm{Trop}^{KP}_{v,\iota}(Y)$ *in w* is defined as

$$\mathrm{mult}(w) \quad = \quad \sum \mathrm{mult}(\mathfrak{p}_i, \mathrm{in}_w(I))$$

where the $\mathfrak{p}_i$ vary through all minimal associated primes of $\mathrm{in}_w(I)$. Note that this multiplicity does neither depend on the choice of section $s : \Gamma \to k^\times$ nor on the choice of primary decomposition $\mathrm{in}_w(I) = \bigcap \mathfrak{q}_i$.

The structure theorem for tropical varieties asserts the following. For details, cf. [37, Thm. 3.3.6].

Theorem 12.1. *The tropicalization of a purely d-dimensional k-subscheme Y of $\mathbb{G}^n_{m,k}$ can be endowed with the structure of a balanced weighted polyhedral complex of dimension d such that the weight of its d-dimensional polyhedra σ equals $\mathrm{mult}(w)$ for each w in the relative interior of σ.*

12.2. Weights from the scheme theoretic tropicalization. Let Y be a purely d-dimensional k-subscheme of $\mathbb{G}^{n,+}_{m,k}$ and Z the associated blue k-scheme. We will show in the following that the weights $\mathrm{mult}(w)$ can be determined from the scheme theoretic tropicalization $\mathrm{Bend}_v(Z)$.

For a more readable notation, we define

$$R_n = k[X_1^{\pm 1}, \dots, X_n^{\pm 1}], \qquad S_n = \mathbb{T}[X_1^{\pm 1}, \dots, X_n^{\pm 1}],$$
$$\overline{R}_n = k_0[X_1^{\pm 1}, \dots, X_n^{\pm 1}], \qquad \overline{S}_n = \mathbb{B}[X_1^{\pm 1}, \dots, X_n^{\pm 1}].$$

Let $I \subset R^+_n$ be the ideal of definition of Y. Then Z is the spectrum of $B = R_n /\!\!/ \mathcal{R}_I$ where $\mathcal{R}_I = \{\sum a_i \equiv \sum b_j | \sum a_i - \sum b_j \in I\}$, and $\mathrm{Bend}_v(Z)$ is the spectrum of $B^\bullet \otimes_{k^\bullet} \mathbb{T} /\!\!/ \mathrm{bend}_v(B)$.

By the definition of B, we can choose a surjection $\pi : S_m \to \mathrm{Bend}_v(B)$, which yields a presentation of the form

$$\mathrm{Bend}_v(B) = S_m /\!\!/ \mathcal{R}_\pi \qquad \text{with} \qquad \mathcal{R}_\pi = \Big\{ \sum a_i \equiv \sum b_j \,\Big|\, \sum \pi(a_i) \equiv \sum \pi(b_j) \Big\}.$$

This corresponds to a closed immersion $\mathrm{Bend}_v(Z) \to \mathbb{G}^m_{m,\mathbb{T}}$. Note that we consider $\mathrm{Trop}_v(Z)$ as an abstract $\mathbb{T}$-scheme; therefore we allow π to be any surjection and m to differ from n.

Consider a point $w \in \mathrm{Bend}_v(Z)(\mathbb{T})$, which is a $\mathbb{T}$-morphism $w : \mathrm{Spec}\,\mathbb{T} \to \mathrm{Bend}_v(Z)$, which corresponds to the morphism $w^* : \mathrm{Bend}_v(B) \to \mathbb{T}$ between the respective blueprints of global sections. Given a polynomial $f = \sum a_e X^e \in S_m^+$ where $a_e \in \mathbb{T}$ and $e \in \mathbb{Z}^m$ is a multi-index, we define $f(w) = \sum a_e w^e$ as $\sum w^*(\pi(a_e X^e))$, which is an element of $\mathbb{T}$. The *initial form of f in w* is

$$\mathrm{in}_w(f) \quad = \quad \sum_{a_e w^e = f(w)} X^e,$$

considered as a polynomial in $\overline{S}_m^+$. Note the formal analogy with the initial form of a polynomial with coefficients in k: the identity valuation $\mathrm{id} : \mathbb{T} \to \mathbb{T}$ yields the *integers* $\mathcal{O}_\mathbb{T} = \{a \in \mathbb{T} | a \leqslant 1\}$, which is a local semiring with maximal ideal $\mathfrak{m}_\mathbb{T} = \{a \in \mathcal{O}_\mathbb{T} | a < 1\}$ and residue field $\mathbb{B} = \mathcal{O}_\mathbb{T}/\mathfrak{m}_\mathbb{T}$. The unique section to id is the identity $\mathrm{id} : \mathbb{T} \to \mathbb{T}$, thus the coefficients $\overline{\mathrm{id}(a_e^{-1}) a_e}$ of the initial form are 1.

The *initial preaddition of $\mathcal{R}_\pi$ in w* is the preaddition

$$\mathrm{in}_w(\mathcal{R}_\pi) \quad \equiv \quad \Big\langle \, \mathrm{in}_w\big(\sum a_i\big) \equiv \mathrm{in}_w\big(\sum b_j\big) \, \Big| \, \sum a_i \equiv \sum b_j \text{ in } \mathcal{R}_\pi \, \Big\rangle$$

on $\overline{S}_m$. Generically, the set $L = \mathrm{Hom}(\overline{S}_m /\!\!/ \mathrm{in}_w(\mathcal{R}_\pi), \mathbb{T})$ is a linear subspace of $(\mathbb{T}^\times)^m \simeq \mathbb{R}^m$. More precisely, L is a linear subspace if w is contained in the open dense subset $\mathcal{U}$ of $\mathrm{Bend}_v(Z)(\mathbb{T})$ that is the union of the interiors of all top-dimensional polyhedras that occur in some realization of $\mathrm{Bend}_v(Z)(\mathbb{T})$ as a polyhedral complex.

If $w \in \mathcal{U}$, then we can exchange $\pi : S_m \to \mathrm{Bend}_v(B)$ by $\pi \circ \varphi$ for a suitable automorphism φ of S_m, so that L coincides with the span $\langle e_1, \dots, e_d \rangle$ of the first d unit vectors of $\mathbb{R}^m$. We define $\mathrm{in}_w(\mathcal{R}_\pi)'$ as the restriction of $\mathrm{in}_w(\mathcal{R}_\pi)$ to $\mathbb{B}[X_{d+1}^{\pm 1}, \dots, X_m^{\pm 1}]$.

A subset S of a $\mathbb{B}$-linear space V is *linearly independent over* $\mathbb{B}$ if every element of V can be written in at most one way as a finite $\mathbb{B}$-linear combination of the elements of S. The *dimension* $\dim_\mathbb{B} V$ *of* V is the supremum of the cardinalities of all linear independent sets.

Definition 12.2. The *Maclagan-Rincón weight of* $w \in \mathcal{U}$ is

$$\mu(w) \quad = \quad \dim_\mathbb{B} \big(\mathbb{B}[X_{d+1}^{\pm 1}, \dots, X_m^{\pm 1}]^+ / \mathrm{in}_w(\mathcal{R}_\pi)' \big).$$

The following theorem is essentially Theorem 1.2 of [36]. Since the weights $\mu(w)$ are not explicitly exhibited in [36] and an additional argument is required, we include a proof.

Theorem 12.3. *The multiplicity* $\mathrm{mult}(w)$ *coincides with the Maclagan-Rincón weight* $\mu(w)$ *for every* $w \in \mathrm{Trop}(Y)$. *In particular,* $\mu(w)$ *is an invariant of* w *and does not depend on the choice of* π.

Proof. As a first point, we observe that the weights of $\mathrm{Trop}(Y)$ are invariant under interchanging coordinates, scaling coordinates and inclusions $\mathbb{T}^n \to \mathbb{T}^m$ as the first n coordinates. This means that the weights of $\mathrm{Trop}(Y) = \mathrm{Bend}_v(Z)(\mathbb{T})$ are invariant under automorphisms of R_n and under a change $Y \to \mathbb{G}^{n,+}_{m,k} \to \mathbb{G}^{m,+}_{m,k}$ of the ambient torus that comes from a morphism $\mathbb{G}^n_{m,k} \to \mathbb{G}^m_{m,k}$ of blue k-scheme.

Therefore we can choose a morphism $\iota : Y \to \mathbb{G}^m_{m,k}$ such that $\iota^+ : Y \to \mathbb{G}^{m,+}_{m,k}$ is a closed immersion of k-schemes and such that $\pi = \mathrm{Bend}_v(\eta)$ where $\eta = \iota^* : R_m \to \Gamma Y$ is the corresponding morphism between the respective global sections. In this case, the preaddition $\mathcal{R}_\pi$ on S_m equals the Giansiracusa bend relation $\mathrm{bend}^{GG}_{v,\iota}(I)$ where $I \subset R_m^+$ is the ideal defining Y. This reduces the proof to the situation of [36].

By our choice of π for a given w, the linear subspace L of $\mathbb{T}^m \simeq \mathbb{R}^m$ equals the span $\langle e_1, \dots, e_d \rangle$ of the first d unit vectors. By [37, Lemma 3.4.7], we have

$$\mathrm{mult}(w) \quad = \quad \dim_{k_0}\big(k_0[X_{d+1}^{\pm 1}, \dots, X_m^{\pm 1}]^+ / \mathrm{in}_w(I)'\big)$$

where $\mathrm{in}_w(I)'$ is the restriction of $\mathrm{in}_w(I)$ to $k_0[X_{d+1}^{\pm1},\dots,X_m^{\pm1}]^+$.

Let $v_0 : k_0 \to \mathbb{B}$ be the trivial valuation. By [36, Prop. 3.4], we have

$$\mathrm{in}_w\big(\mathrm{bend}^{GG}_{v,\eta}(I)\big) \quad = \quad \mathrm{bend}^{GG}_{v_0,\eta}\big(\mathrm{in}_w(I)\big).$$

Both in_w and $\mathrm{bend}^{GG}_{v,\eta}$ commute with the restriction to the variables $X_{d+1},\dots,X_m$. Therefore we obtain $\mathrm{in}_w\big(\mathrm{bend}^{GG}_{v,\eta}(I)\big)' = \mathrm{bend}^{GG}_{v_0,\eta}\big(\mathrm{in}_w(I)'\big)$ and

$$\mathbb{B}[X_{d+1}^{\pm1},\dots,X_m^{\pm1}]^+ \,/\, \mathrm{in}_w\big(\mathrm{bend}^{GG}_{v,\eta}(I)\big)' \quad = \quad \mathrm{Trop}^{GG}_{v_0,\eta}\big(k_0[X_{d+1}^{\pm1},\dots,X_m^{\pm1}]^+/\,\mathrm{in}_w(I)'\big).$$

Since $k_0[X_{d+1}^{\pm1},\dots,X_m^{\pm1}]^+/\mathrm{in}_w(I)'$ is a k_0-vector space of finite dimension $\mathrm{mult}(w)$, its tropicalization is a $\mathbb{B}$-linear space of the same dimension, cf. [21, Lemma 7.1.3]. Since $\mathcal{R}_\pi = \mathrm{bend}^{GG}_{v,\iota}(I)$, this dimension equals, by definition, the Maclagan-Rincón weight $\mu(w)$. This shows that $\mu(w) = \mathrm{mult}(w)$ and finishes the proof. □

Theorem 12.3 allows us to apply the structure theorem for tropicalizations to the scheme theoretic tropicalization $\mathrm{Bend}_v(Z)$ of the blue k-scheme Z associated with a purely d-dimensional closed subscheme Y of $\mathbb{G}^{n,+}_{m,k}$.

Corollary 12.4. *Let $\mathrm{Bend}_v(Z) \to \mathbb{G}^m_{m,\mathbb{T}}$ be a closed immersion of blue $\mathbb{T}$-schemes. Then we can endow $\mathrm{Bend}_v(Z)(\mathbb{T}) \subset (\mathbb{T}^\times)^m \simeq \mathbb{R}^m$ with the structure of a balanced weighted polyhedral complex of dimension d such that the weight of its d-dimensional polyhedra σ equals $\mu(w)$ for each w in the relative interior of σ.* □

12.3. What is new? In this section, we have extended the results from [36] to the intrinsic tropicalization of a closed subvariety of a torus as a blue $\mathbb{T}$-scheme. This is a subtle step towards a more rigorous setting of scheme theory for tropical geometry.

Moreover, we have exhibited an explicit formula of the Maclagan-Rincón weights, which opens the door to investigate weights in more general situations, for example for valuations $v : k \to \mathbb{T}$ whose image is not dense or even trivial, or in the case of blue $\mathbb{T}$-schemes that are not tropicalizations of classical varieties.

13. Macpherson analytification

Let k be a ring and A a k-algebra. One of the key ideas of Macpherson's paper [38] is that the semiring $\mathrm{An}(A,k)$ of finitely generated k-submodules of A represents the functor of valuations on A in idempotent semirings that are integral on k. The focus of [38] lies on extending this concept to non-Archimedean analytic geometry, for which reason the pair (A,k) is assumed to form a non-Archimedean ring. We refrain from an excursion into non-Archimedean geometry, but we will extend $\mathrm{An}(A,k)$ to ordered blueprints k and A.

13.1. The universal Giansiracusa tropicalization as Macpherson analytification. Let us begin with reviewing the results and immediate implications of [38]. The *Macpherson analytification* $\mathrm{An}(A,k)$ is the semiring of all k-submodules of A with respect to the addition

$$M_1 + M_2 \quad = \quad \big\{\, m \in A \,\big|\, m = m_1 + m_2 \text{ for some } m_1 \in M_1 \text{ and } m_2 \in M_2 \,\big\}$$

and the multiplication

$$M_1 \cdot M_2 \quad = \quad \big\{\, m \in A \,\big|\, m = m_1 \cdot m_2 \text{ for some } m_1 \in M_1 \text{ and } m_2 \in M_2 \,\big\}.$$

The semiring $\mathrm{An}(A,k)$ is idempotent and comes with the valuation $v : A \to \mathrm{An}(A,k)$, which sends $a \in A$ to the k-submodule of A generated by a. This valuation is *integral on k*, i.e. $v(a) + 1 = 1$ for all $a \in k$.

Let $\mathrm{Val}(A,k;-) : \mathrm{Alg}_{\mathbb{B}} \to \mathrm{Sets}$ be the functor of all valuations on A in idempotent semirings that are integral on k. The key observation of Macpherson is that $\mathrm{An}(A,k)$ represents $\mathrm{Val}(A,k;-)$.

From this, we can deduce the following description of the *universal Giansiracusa tropicalization* $\mathrm{Trop}^{GG}_{v,\eta}(A)$ where $v : k \to T$ is a valuation into a totally ordered idempotent semiring T and $\eta : k[A^\bullet] \to A$ is the natural k-linear map. Let

$$\mathcal{O}_k \quad = \quad \big\{ a \in k \,\big|\, v(a)+1 = 1 \big\}$$

be the subring of integral elements in k. Then the valuation $v : k \to T$ corresponds to a homomorphism $\mathrm{An}(k,\mathcal{O}_k) \to T$ of semirings by the universal property of $\mathrm{An}(k,\mathcal{O}_k)$. Since $\mathrm{Trop}^{GG}_{v,\eta}(A)$ represents $\mathrm{Val}_v(A,-)$ and $\mathrm{Val}_v(A,S)$ corresponds to the valuations w in $\mathrm{Val}(A,\mathcal{O}_k;S)$ that restrict to $w|_k = v$, we have a canonical isomorphism of semirings

$$\mathrm{Trop}^{GG}_{v,\eta}(A) \quad \stackrel{\sim}{\longrightarrow} \quad \mathrm{An}(A,\mathcal{O}_k) \otimes^+_{\mathrm{An}(k,\mathcal{O}_k)} T$$

where $B \otimes^+_D C$ stays for $(B \otimes_D C)^+$, cf. section 1.

By glueing affine patches, this connection generalizes to a description of the universal Giansiracusa tropicalization of a k-scheme X. A variation of the definition of $\mathrm{An}(A,k)$ yields a description of the Giansiracusa tropicalization for a closed immersion $\iota : Y \to X$ into a toric variety, see [38, para. 7.3].

In the following, we will make this precise by different means: we extend the Macpherson analytification to all ordered blueprints k and A, which yields a description of the bend functor in terms of $\mathrm{An}(A,k)$ and, conversely, a description of $\mathrm{An}(A,k)$ as bend.

13.2. The Macpherson analytification as bend. Let k be an ordered blueprint and B an ordered blue k-algebra. A *k-span in B* is a subset M of B that is closed under multiplication by elements of k and that contains all $a \in B$ for which there are $b_j \in M$ such that $a \leqslant \sum b_j$. We write $\langle a_i \rangle$ for the smallest k-span in B that contains the elements a_i. A k-span M in B is *finitely generated* if $M = \langle a_i \rangle$ for finitely many elements $a_i \in B$. For $a \in k$, we write $\langle a \rangle = \langle \bar{a} \rangle$ where $\bar{a}$ is the image of a in B.

The semiring $\mathrm{An}(B,k)$ is the set of all finitely generated k-spans in B together with the addition

$$M_1 + M_2 \quad = \quad \big\{ m \in B \,\big|\, m \leqslant m_1 + m_2 \text{ for some } m_1 \in M_1 \text{ and } m_2 \in M_2 \big\}$$

and the multiplication

$$M_1 \cdot M_2 \quad = \quad \big\{ m \in B \,\big|\, m = m_1 \cdot m_2 \text{ for some } m_1 \in M_1 \text{ and } m_2 \in M_2 \big\}.$$

Note that this recovers the definition of $\mathrm{An}(B,k)$ in the case of rings k and B, and that $\mathrm{An}(B,k)$ is an idempotent semiring for all ordered blueprints k and B. In other words, $\mathrm{An}(B,k)$ is a $\mathbb{B}$-algebra.

The $\mathbb{B}$-algebra $\mathrm{An}(B,k)$ comes with the map $v : B \to \mathrm{An}(B,k)$ that sends a to $\langle a \rangle$. This map is a morphism between the underlying monoids. If $a \leqslant \sum b_j$ in B^{mon}, then $\langle a \rangle \subset \langle b_j \rangle$ as subsets of B, which implies that $\langle a \rangle + \sum \langle b_j \rangle = \sum \langle b_j \rangle$ in $\mathrm{An}(B,k)$. Thus $\langle a \rangle \leqslant \sum \langle b_j \rangle$ in $\mathrm{An}(B,k)^{\mathrm{pos}}$. This shows that $v : B \to \mathrm{An}(B,k)$ is a valuation.

For $a \in k$, we have $\langle a \rangle \subset \langle 1 \rangle$ as subsets of B and thus $\langle a \rangle + \langle 1 \rangle = \langle 1 \rangle$ in $\mathrm{An}(B,k)$. Therefore $v(a) \leqslant 1$ in $\mathrm{An}(B,k)^{\mathrm{pos}}$, which means, by definition, that v is integral on k.

Lemma 13.1. *Let S be a multiplicative subset of B. Then the association $\langle \frac{a}{s} \rangle \mapsto \frac{\langle a \rangle}{\langle s \rangle}$ defines an isomorphism* $\mathrm{An}(S^{-1}B,k) \to v(S)^{-1}\,\mathrm{An}(B,k)$.

Proof. Since $s \in S$ is mapped to the invertible element $\langle s\rangle$ of $v(S)^{-1}\operatorname{An}(B,k)$, the association $\langle \frac{a}{s}\rangle \mapsto \frac{\langle a\rangle}{\langle s\rangle}$ defines a morphism $\operatorname{An}(S^{-1}B,k) \to v(S)^{-1}\operatorname{An}(B,k)$. Conversely, $\langle s\rangle \in v(S)$ is invertible in $\operatorname{An}(S^{-1}B,k)$. Thus the inverse association $\frac{\langle a\rangle}{\langle s\rangle} \mapsto \langle \frac{a}{s}\rangle$ defines an inverse morphism $v(S)^{-1}\operatorname{An}(B,k) \to \operatorname{An}(S^{-1}B,k)$. □

As a consequence of this lemma, an affine presentation $\mathcal{U}$ of an ordered blue k-scheme X yields an affine presentation $\operatorname{An}(\mathcal{U},k)$ in $\mathbb{B}$-algebras. We define $\operatorname{An}(X,k)$ as the colimit of $\operatorname{An}(\mathcal{U},k)$. It comes with a valuation $v : \operatorname{An}(X,k) \to X$ that is integral on k, which means that for all affine open subschemes $U = \operatorname{Spec} B$ of X, the induced valuation $\Gamma v|_U : B \to \operatorname{An}(B,k)$ is integral on k. We denote by $\operatorname{Val}(X,k;-)$ the functor that takes a $\mathbb{B}$-algebra S to the set of valuations $\operatorname{Spec} S \to X$ that are integral on k.

Let B be an ordered blue k-algebra. We define

$$B^{\mathrm{mon}}_{k\leqslant 1} \quad = \quad B^{\mathrm{mon}} /\!\!/ \langle \bar{a} \leqslant 1 | a \in k\rangle$$

where $\bar{a}$ is the image of a in B. This definition is obviously invariant under localization. Thus we can define for an ordered blue k-scheme X with affine presentation $\mathcal{U}$ the affine presentation $\mathcal{U}^{\mathrm{mon}}_{k\leqslant 1}$, and $X^{\mathrm{mon}}_{k\leqslant 1}$ as its colimit.

Theorem 13.2. *Let k be an ordered blueprint, X an ordered blue k-scheme and $v : \mathbb{F}_1 \to \mathbb{B}$ the trivial valuation. Then there is a canonical morphism*

$$\operatorname{An}(X,k) \quad \longrightarrow \quad \operatorname{Bend}_v(X^{\mathrm{mon}}_{k\leqslant 1})$$

that induces an isomorphism $\operatorname{An}(X,k) \simeq \operatorname{Bend}_v(X^{\mathrm{mon}}_{k\leqslant 1})^+$ *of semiring schemes, and* $\operatorname{An}(X,k)$ *represents* $\operatorname{Val}(X,k;-)$.

Proof. Since all constructions in questions are defined in terms of affine presentations, it is enough to prove the theorem in the affine case $X = \operatorname{Spec} B$. In this case, the canonical map

$$\psi : \operatorname{Bend}_v(B^{\mathrm{mon}}_{k\leqslant 1}) \;=\; B^{\bullet} \otimes_{\mathbb{F}_1} \mathbb{B} /\!\!/ \operatorname{bend}_v(B^{\mathrm{mon}}_{k\leqslant 1}) \quad \longrightarrow \quad \operatorname{An}(B,k)$$

is given by $a \otimes 1 \mapsto \langle a\rangle$. This map is clearly a morphism between the respective underlying monoids, and since $\langle a\rangle + \langle a\rangle = \langle a\rangle$, it is $\mathbb{B}$-linear. In order to see that ψ respects the bend relations, consider a relation $a \leqslant \sum b_j$ in $B^{\mathrm{mon}}_{k\leqslant 1}$. Then $\langle a\rangle \subset \langle b_j\rangle$ and thus $\langle a, b_j\rangle = \langle b_j\rangle$. Thus the relation $a + \sum b_j \equiv \sum b_j$ in $\operatorname{bend}_v(B^{\mathrm{mon}}_{k\leqslant 1})$ implies that $\langle a\rangle + \sum\langle b_j\rangle \equiv \sum\langle b_j\rangle$ in $\operatorname{An}(B,k)$. This shows that ψ is a morphism of ordered blueprints.

Since $\operatorname{An}(B,k)$ is generated by the principal ideals $\langle a\rangle$ as a semiring, it suffices to show that every relation in $\operatorname{An}(B,k)$ is already contained in $\operatorname{Bend}_v(B^{\mathrm{mon}}_{k\leqslant 1})$ in order to prove that $\psi^+ : \operatorname{Bend}_v(B^{\mathrm{mon}}_{k\leqslant 1})^+ \to \operatorname{An}(B,k)$ is an isomorphism. Therefore, let us consider an equality $\langle a_i\rangle = \langle b_j\rangle$ of k-spans of B. Then we have for all i a relation $a_i \leqslant \sum c_{i,j} b_j$ for certain $c_{i,j} \in k$ and for all j a relation $b_j \leqslant \sum d_{j,i} a_i$ for certain $d_{j,i} \in k$. Since in $B^{\mathrm{mon}}_{k\leqslant 1}$, we have $c_{i,j} \leqslant 1$ and $d_{j,i} \leqslant 1$, this implies $a_i \leqslant \sum b_j$ and $b_j \leqslant \sum a_i$ in $B^{\mathrm{mon}}_{k\leqslant 1}$. Therefore we find the relations $a_i + \sum b_j \equiv \sum b_j$ and $\sum a_i \equiv \sum a_i + b_j$ in $\operatorname{bend}_v(B^{\mathrm{mon}}_{k\leqslant 1})$. Using that $\operatorname{Bend}_v(B^{\mathrm{mon}}_{k\leqslant 1})$ is idempotent, we find

$$\sum_i a_i \;=\; \sum_{j,i} a_i \;=\; \sum_j \Big(\sum_i a_i + b_j\Big) \;=\; \sum_i a_i + \sum_j b_j \;=\; \sum_i \Big(a_i + \sum_j b_j\Big) \;=\; \sum_{i,j} b_j \;=\; \sum b_j$$

in $\operatorname{Bend}_v(B^{\mathrm{mon}}_{k\leqslant 1})$. This shows that ψ^+ is an isomorphism of semirings.

By Theorem 7.16, we know that $\operatorname{Bend}_v(B^{\mathrm{mon}}_{k\leqslant 1})$ represents the functor $\operatorname{Val}_v(B^{\mathrm{mon}}_{k\leqslant 1},-)$ on ordered blue $\mathbb{B}$-algebras. Since ψ^+ is an isomorphism of semirings, $\operatorname{An}(B,k)$ represents the restriction $\operatorname{Val}^+_v(B^{\mathrm{mon}}_{k\leqslant 1},-)$ of $\operatorname{Val}_v(B^{\mathrm{mon}}_{k\leqslant 1},-)$ to $\mathbb{B}$-algebras. A valuation $w : B \to S$ in a $\mathbb{B}$-algebra S that is integral on k is the same as a valuation $w : B^{\mathrm{mon}}_{k\leqslant 1} \to S$, and every valuation in a $\mathbb{B}$-algebra is an extension of the trivial valuation $v : \mathbb{F}_1 \to \mathbb{B}$. Therefore the functors $\operatorname{Val}^+_v(B^{\mathrm{mon}}_{k\leqslant 1},-)$ and

$\mathrm{Val}(B,k;-)$ are isomorphic. We conclude that $\mathrm{An}(B,k)$ represents $\mathrm{Val}(B,k;-)$, which completes the proof of the theorem. $\square$

Remark 13.3. Note that for a description of the Macpherson analytification, we use non-algebraic blueprints in an essential way. Though the bends $\mathrm{Bend}_v(B^{\mathrm{mon}})$ and $\mathrm{Bend}_v(B)$ coincide, the relation $a \leqslant 1$ for $a \in k$ implies $a = 1$ in $B_{k\leqslant 1}$ in the typical case that B is with -1. Moreover, we have to endow B with the relations $a \leqslant 1$ for $a \in k$ to guarantee that the bend represents only valuations that are integral on k.

Corollary 13.4. *Let k be an ordered blueprint, X an ordered blue k-scheme and $v : k \to T$ be a valuation in an idempotent semiring. Let $\mathcal{O}_k = \{a \in k | v(a) \leqslant 1 \text{ in } T^{\mathrm{pos}}\}$. Then there exists a canonical isomorphism*

$$\mathrm{Bend}_v(X)^+ \quad \xrightarrow{\sim} \quad \mathrm{An}(X,\mathcal{O}_k) \otimes^+_{\mathrm{An}(k,\mathcal{O}_k)} T.$$

Proof. Similar to our argument in section 13.1, this can be proven by showing that both semirings represent the functor $\mathrm{Val}(X,k;-)$. An alternative proof is the following direct calculation.

Let $v_0 : \mathbb{F}_1 \to \mathbb{B}$ be the trivial valuation and C an ordered blue blueprint. Then we have

$$\mathrm{Bend}_{v_0}(C) \quad = \quad C^\bullet \otimes_{\mathbb{F}_1} \mathbb{B} /\!\!/ \mathrm{bend}_{v_0}(C) \quad = \quad C /\!\!/ \mathrm{bend}(C)$$

where $\mathrm{bend}(C) = \{a + \sum b_j \equiv \sum b_j | a \leqslant \sum b_j \text{ in } C\}$ does not depend on the valuation v_0. Therefore, we obtain

$$\begin{aligned} \mathrm{Bend}_{v_0}(B^{\mathrm{mon}}_{\mathcal{O}_k\leqslant 1}) \otimes_{\mathrm{Bend}_{v_0}(k^{\mathrm{mon}}_{\mathcal{O}_k\leqslant 1})} T \quad &= \quad B^\bullet /\!\!/ \mathrm{bend}(B^{\mathrm{mon}}_{\mathcal{O}_k\leqslant 1}) \otimes_{k^\bullet /\!\!/ \mathrm{bend}(k^{\mathrm{mon}}_{\mathcal{O}_k\leqslant 1})} T \\ &= \quad B^\bullet \otimes_{k^\bullet} T /\!\!/ \mathrm{bend}_v(B^{\mathrm{mon}}_{\mathcal{O}_k\leqslant 1}). \end{aligned}$$

Since, by definition of $\mathcal{O}_k$, an element $a \in \mathcal{O}_k$ implies the bend relation $a + 1 \equiv 1$ in T, we conclude that this ordered blueprint is isomorphic to

$$B^\bullet \otimes_{k^\bullet} T /\!\!/ \mathrm{bend}_v(B^{\mathrm{mon}}) \quad = B^\bullet \otimes_{k^\bullet} T /\!\!/ \mathrm{bend}_v(B) \quad = \quad \mathrm{Bend}_v(B).$$

The claim of the Corollary follows from applying $(-)^+$ to the above equations and using the isomorphism $\mathrm{An}(C,\mathcal{O}_k) \simeq \mathrm{Bend}_{v_0}(C_{\mathcal{O}_k\leqslant 0})^+$ from Theorem 13.2. $\square$

Together with Theorem 11.2, this yields the following description of the Giansiracusa tropicalization in its general form.

Corollary 13.5. *Let k be a ring, $v : k \to T$ be a valuation in a totally ordered idempotent semiring T. Let $\iota : X \to Y$ a morphism from a k-scheme X into a monoid scheme Y such that $\iota^+ : X \to Y_k^+$ is a closed immersion. Then there is a canonical isomorphism*

$$\mathrm{Trop}^{GG}_{v,\iota}(Y) \quad \xrightarrow{\sim} \quad \mathrm{An}(Z,\mathcal{O}_k) \otimes^+_{\mathrm{An}(k,\mathcal{O}_k)} T$$

where Z is the associated blue scheme and $\mathcal{O}_k = \{a \in k | v(a) \leqslant 1 \text{ in } T^{\mathrm{pos}}\}$. $\square$

14. Thuillier analytification

An important variant of the Berkovich analytification was found by Thuillier in [50] in case of the trivial valuation $v : k \to \mathbb{T}$ of a field k. In the following, we will review the definition of $X^\beth$, and reinterpret this topological space in terms of the scheme theoretic tropicalization of X.

Let $\mathcal{O}_\mathbb{T} = \{a \in \mathbb{T} | a + 1 = 1\}$ be the subsemiring of $\mathbb{T}$ whose underlying set is the real interval $[0,1]$. We endow $\mathcal{O}_\mathbb{T}$ with the real topology of $[0,1] \subset \mathbb{R}$.

In the affine case $X = \operatorname{Spec} B$, the *Thuillier analytification* $X^\beth$ consists of all valuations $w : B \to \mathbb{T}$ in X^{an} whose image is contained in $\mathcal{O}_\mathbb{T}$. It comes with the subspace topology of X^{an}. In the case of an arbitrary k-scheme X with affine presentation $\mathcal{U}$, we define $X^\beth$ as the colimit of $\mathcal{U}^\beth$ as a topological space.

Theorem 14.1. *The Thuillier space $X^{\beth}$ is naturally homeomorphic to* $\mathrm{Bend}_v(X)(\mathcal{O}_{\mathbb{T}})$.

Proof. Note that the semiring $\mathcal{O}_{\mathbb{T}}$ is a local topological Hausdorff semiring with open unit group. This allows us to apply the same proof as for Theorem 8.1 to verify the claim of the theorem. $\square$

Remark 14.2. An alternative realization of $X^{\beth}$ as a rational point set is the following. Define $B^{\mathrm{mon}}_{\leqslant 1} = B^{\mathrm{mon}} /\!\!/ \langle a \leqslant 1 | a \in B\rangle$ for an ordered blueprint B. Then every valuation $w : B^{\mathrm{mon}}_{\leqslant 1} \to \mathbb{T}$, which is a morphism $\tilde{w} : B^{\mathrm{mon}}_{\leqslant 1} \to \mathbb{T}^{\mathrm{pos}}$, respects the relation $a \leqslant 1$, i.e. $w(a) \in \mathcal{O}_{\mathbb{T}}$. If we define $X^{\mathrm{mon}}_{\leqslant 1}$ in terms of an affine presentation, then every valuation $\omega : \mathrm{Spec}\,\mathbb{T} \to X^{\mathrm{mon}}_{\leqslant 1}$ factors through the morphism $\mathrm{Spec}\,\mathbb{T} \to \mathrm{Spec}\,\mathcal{O}_{\mathbb{T}}$ that corresponds to the inclusion $\mathcal{O}_{\mathbb{T}} \subset \mathbb{T}$. We conclude that we have natural bijections

$$X^{\beth} \quad = \quad \mathrm{Val}_v(X^{\mathrm{mon}}_{\leqslant 1}, \mathcal{O}_{\mathbb{T}}) \quad = \quad \mathrm{Val}_v(X^{\mathrm{mon}}_{\leqslant 1}, \mathbb{T}) \quad = \quad \mathrm{Bend}_v(X^{\mathrm{mon}}_{\leqslant 1})(\mathbb{T}),$$

which yields, in fact, a homeomorphism of topological spaces.

15. Ulirsch tropicalization

In case of a field k with trivial valuation $v : k \to \mathbb{T}$, and a toroidal embedding $U \subset X$ without self-intersection, Thuillier defines in [50] a retraction $X^{\beth} \to \overline{\Sigma}_X$ onto an extended cone complex $\overline{\Sigma}_X$ associated with $U \subset X$. Abramovich, Caporaso and Payne interpret in [1] this retraction map as the tropicalization of X. Ulirsch generalizes in [53] Thuillier's tropicalization by associating a log structure to a toroidal embedding. Ulirsch's tropicalization passes through an associated Kato fan that allows him to apply the local tropicalization of Popescu-Pampu and Stepanov in [47] to charts of the log structure. This also recovers the tropicalization of fine and saturated log schemes as studied by Gross and Siebert in [23].

In this section, we will review Ulirsch's tropicalization of log schemes and connect it to the scheme theoretic tropicalization developed in this paper, under some additional assumptions: we restrict ourselves to Zariski log structures and require that the log scheme has an affine open covering that is compatible with the log-structure; cf. Remark 15.2 and section 15.6. In contrast to [53], we write all monoids multiplicatively.

15.1. Kato fans. A *monoidal space* is a topological space X together with a sheaf of monoids $\mathcal{M}_X$. Note that every monoid M is local since the complement of its units $M^{\times}$ forms the unique maximal ideal of the monoid. Therefore every stalk $\mathcal{M}_{X,x}$ is a local monoid with maximal ideal $\mathfrak{m}_x$. A *(local) morphism of monoidal spaces* is a continuous map $\varphi : X \to Y$ together with a morphism $\varphi^{\flat} : \varphi^{-1}\mathcal{M}_Y \to \mathcal{M}_X$ of sheaves of monoids such that the induced morphisms of stalks $\varphi_x : \mathcal{M}_{Y,y} \to \mathcal{M}_{X,x}$ map $\mathfrak{m}_y$ to $\mathfrak{m}_x$ for all $x \in X$ and $y = \varphi(x)$. A morphism $\varphi : X \to Y$ of monoidal spaces is *strict* if $\varphi^{\flat} : \varphi^{-1}\mathcal{M}_Y \to \mathcal{M}_X$ is an isomorphism.

The unit group $M^{\times}$ acts by multiplication on M, and the quotient $M/M^{\times}$ of this action inherits the structure of a monoid since multiplication is commutative.

A monoid is *sharp* if $M^{\times} = \{1\}$. A monoidal space X is *sharp* if $\mathcal{M}_X(U)^{\times} = \{1\}$ for all open subsets U of X. Given a monoidal space $(X, \mathcal{M}_X)$, we define its associated sharp monoidal space as $(X, \overline{\mathcal{M}}_X)$ where $\overline{\mathcal{M}}_X = \mathcal{M}_X / \mathcal{M}_X^{\times}$.

A *multiplicative set in M* is a multiplicatively closed subset $S \subset M$ that contains 1. The *localization of M at S* is $S^{-1}M = S \times M / \sim$ where $(s,m) \sim (s',m')$ if and only if there is a $t \in S$ such that $tsm' = ts'm$. We denote the class of (s,m) in $S^{-1}M$ by $\frac{m}{s}$. A *prime ideal of M* is a subset $\mathfrak{p} \subset M$ such that $\mathfrak{p}M = \mathfrak{p}$ and $S = M - \mathfrak{p}$ is a multiplicative set.

The *affine Kato fan* $\mathrm{Spec}^K M$ of a monoid M is the following sharp monoidal space. Its underlying topological space is the set of all prime ideals $\mathfrak{p}$ of M endowed with the topology generated by open subsets of the form $U_h = \{\mathfrak{p} | h \notin \mathfrak{p}\}$. Its structure sheaf $\mathcal{M}_{\mathrm{Spec}^K M}$ associates with U_h the sharp monoid $S^{-1}M/(S^{-1}M)^{\times}$ where $S = \{h^i\}_{i \geqslant 0}$. A *Kato fan* is a sharp monoidal space that has an open covering by affine Kato fans.

A monoid M is *fine* if it is finitely generated and embeds into its Grothendieck group M^{gp}. A fine monoid is *saturated* if $a^n \in M$ with $a \in M^{\mathrm{gp}}$ and $n \geqslant 1$ implies $a \in M$. A Kato fan is *fine and saturated* if it can be covered by affine Kato fans of fine and saturated monoids.

Remark 15.1. Some of the notions introduced in this section have already been defined for monoids with zero, which are ordered blueprints. Though most notions are in spirit the same and can be recovered by associating an additional element 0 to a monoid in the sense of this section, there is an important digression in the notion of the spectrum. While the spectrum of a monoid A with zero associates with an open U_h localizations $S^{-1}A$ where $S = \{h^i\}_{i\geqslant 0}$, the affine Kato fan of a monoid M associates with an open U_h the sharp monoid $S^{-1}M/(S^{-1}M)^\times$.

15.2. Extended cone complexes. Consider the intervals $S = (0,1]$ and $S_0 = [0,1]$, which are both multiplicative monoids endowed with the real topology. Let M be a fine and saturated monoid. The *cone of* M is the homomorphism set $\sigma_M = \mathrm{Hom}(M,S)$ endowed with the compact-open topology where we regard M as a discrete monoid. Note that σ_M is indeed a cone in the $\mathbb{R}$-vector space $\mathrm{Hom}(M^{\mathrm{gp}},\mathbb{R}_{>0})$ where $\mathbb{R}$ acts on $\mathbb{R}_{>0}$ via the exponential map.

Let F be a fine and saturated Kato fan and $V' \subset V$ open affine Kato subfans where $V' = \mathrm{Spec}^K M'$ and $V = \mathrm{Spec}^K M$ for some fine, saturated and sharp monoids M' and M. Then $M' = S^{-1}M/(S^{-1}M)^\times$ for some multiplicative set S in M, which implies that $\sigma_{M'}$ is a face of σ_M. Let Δ be the diagram of all cones σ_M such that $\mathrm{Spec}^K M$ is an affine open in F together with the face maps $\sigma_{M'} \subset \sigma_M$ for which $\mathrm{Spec}^K M' \subset \mathrm{Spec}^K M$. We define the *cone complex* Σ_F *of* F as the colimit of Δ as a topological space. As a point set Σ_F is equal to $\mathrm{Hom}(\mathrm{Spec}^K S, F)$.

Similarly, we define the *extended cone of* M as $\overline{\sigma}_M = \mathrm{Hom}(M,S_0)$ endowed with the compact-open topology. The affine Kato subfans of F yield a diagram $\overline{\Delta}$ of extended cones and face maps. We define the *extended cone complex* $\overline{\Sigma}_F$ *of* F as the colimit of $\overline{\Delta}$ as a topological space. As a point set $\overline{\Sigma}_F$ is equal to $\mathrm{Hom}(\mathrm{Spec}^K S_0, F)$.

15.3. Log schemes. Let X be a k-scheme. A *pre-logarithmic structure for* X is a sheaf of monoids $\mathcal{M}_X$ on X together with a morphism $\alpha : \mathcal{M}_X \to \mathcal{O}_X$ of sheaves of monoids where the structure sheaf $\mathcal{O}_X$ is regarded as a sheaf of monoids with respect to multiplication. A *logarithmic structure for* X is a pre-logarithmic structure $\mathcal{M}_X$ such that $\alpha : \mathcal{M}_X \to \mathcal{O}_X$ induces an isomorphism $\alpha^{-1}\mathcal{O}_X^\times \to \mathcal{O}_X^\times$. One can associate to every pre-logarithmic structure $\mathcal{M}_X$ the logarithmic structure $\mathcal{M}_X^a$ that is the push-out of the diagram

$$\begin{tikzcd} \alpha^{-1}\mathcal{O}_X^\times \arrow[r] \arrow[d, "\alpha"'] & \mathcal{M}_X \\ \mathcal{O}_X^\times & \end{tikzcd}$$

in the category of sheaves in monoids.

A *log scheme* is a scheme X together with a logarithmic structure $\mathcal{M}_X$. Given a morphism $\varphi : X \to Y$ of schemes, we define the *inverse image* $\varphi^*\mathcal{M}_X$ of a logarithmic structure $\mathcal{M}_Y$ on Y as the logarithmic structure associated with $\varphi^{-1}\mathcal{M}_Y$.

A *morphism of log schemes* is a morphism $\varphi : X \to Y$ of schemes together with a morphism $\varphi^\flat : \varphi^*\mathcal{M}_Y \to \mathcal{M}_X$ of sheaves of monoids such that

$$\begin{tikzcd} \varphi^*\mathcal{M}_Y \arrow[r, "\varphi^\flat"] \arrow[d, "\varphi^*(\alpha_Y)"'] & \mathcal{M}_X \arrow[d, "\alpha"] \\ \varphi^{-1}\mathcal{O}_Y \arrow[r, "\varphi^\sharp"] & \mathcal{O}_X \end{tikzcd}$$

commutes.

Let X be a log scheme. Given a monoid M, we denote by M_X the constant sheaf with value M. A *chart for* X is a morphism $\beta : M_X \to \mathcal{M}_X$ of sheaves of monoids such that $\beta^a : M_X^a \to \mathcal{M}_X$

is an isomorphism. A log scheme X is called *fine and saturated* if it admits a covering by open subschemes U_i with charts $\beta_i : (M_i)_{U_i} \to \mathcal{M}_{U_i}$ for fine and saturated monoids M_i where $\mathcal{M}_{U_i}$ denotes the restriction of $\mathcal{M}_X$ to U_i.

Remark 15.2. In this exposition, we restrict ourselves to Zariski log schemes, for which we can make the connection to the scheme theoretic tropicalization precise, and we leave an extension of the theory to étale structures to future investigations.

15.4. The associated Kato fan. Recall that $\overline{\mathcal{M}}_X = \mathcal{M}_X/\mathcal{M}_X^\times$ is the sharp sheaf of monoids associated with $\mathcal{M}_X$. A log structure $\alpha : \mathcal{M}_X \to X$ is said to be *without monodromy* if there exists a Kato fan F and a strict morphism $(X, \overline{\mathcal{M}}_X) \to F$. This is, for instance, the case if $\alpha : \mathcal{M}_X \to \mathcal{O}_X$ is injective. See [23, Ex. B.1] and [53, Ex. 4.12] for examples of log structures with monodromy.

The following is the key observation that allows us to define the tropicalization of a fine and saturated log scheme. This is Proposition 4.7 in [53], though it is essentially already present in [27].

Proposition 15.3. *Let X be a scheme of finite type over k and $\alpha : \mathcal{M}_X \to \mathcal{O}_X$ a fine and saturated log structure without monodromy . Then there is a strict morphism $(X, \overline{\mathcal{M}}_X) \to F_X$ of sharp monoidal spaces into a Kato fan F_X that is initial for all strict morphisms from $(X, \overline{\mathcal{M}}_X)$ into a Kato fan.*

We call F_X the *associated Kato fan of X* and the composition

$$\chi_X : (X, \overline{\mathcal{O}}_X) \quad \longrightarrow \quad (X, \overline{\mathcal{M}}_X) \quad \longrightarrow \quad F_X$$

its *characteristic morphism* where $\overline{\mathcal{O}}_X = \mathcal{O}_X/\mathcal{O}_X^\times$. We briefly write $\chi_X : X \to F_X$ for the characteristic morphism.

15.5. Tropicalization of a fine and saturated log scheme. Let X be a fine and saturated log scheme over k with log structure $\alpha : \mathcal{M}_X \to \mathcal{O}_X$ and characteristic morphism $\chi_X : X \to F_X$. The *Ulirsch tropicalization* $\mathrm{Trop}^U_\alpha(X)$ *of X* is the extended cone complex $\overline{\Sigma}_X = \overline{\Sigma}_{F_X}$ together with a surjective continuous map $\mathrm{trop}^U_\alpha : X^\beth \to \overline{\Sigma}_X$ that is described for affine open subsets as follows.

Let $U = \operatorname{Spec} R$ be open in X and $V = \operatorname{Spec}^K M$ open in F_X such that $\chi_X(U) \subset V$ and $M_U \to \mathcal{M}_U$ is a chart. By Lemma 1.6 in [27], X and F_X can be covered by such open subsets. This yields a morphism of sheaves of monoids $M_U \to \mathcal{M}_U \to \overline{\mathcal{O}}_U$ and, by taking global sections, a multiplicative map $M \to R/R^\times$. Since every valuation $w : R \to \mathcal{O}_\mathbb{T}$ maps $R^\times$ to 1 and since S_0 is the underlying monoid of $\mathcal{O}_\mathbb{T}$, we get a continuous map

$$\mathrm{trop}^U_\alpha : U^\beth = \mathrm{Val}_v(R, \mathcal{O}_\mathbb{T}) \quad \longrightarrow \quad \mathrm{Hom}(M, S_0) = \overline{\Sigma}_{\operatorname{Spec}^K M}.$$

If $\iota : Y \to X$ is a closed immersion of k-schemes, then the *Ulirsch tropicalization of Y with respect to ι* is the image $\mathrm{Trop}^U_{\alpha,\iota}(Y) = \mathrm{trop}^U_\alpha(Y)$ of Y in $\mathrm{Trop}^U_\alpha(X)$, together with the restriction $\mathrm{trop}^U_{\alpha,\iota} : Y^\beth \to \mathrm{Trop}^U(Y)$ of trop^U_α to $Y^\beth$.

15.6. The associated blue scheme. Let X be a fine and saturated log scheme X over k without monodromy and $\chi_X : X \to F_X$ the characteristic map into the associated Kato fan F_X. Provided that the inverse images $U = \chi^{-1}(V)$ of affine open Kato subfans V of F_X are affine, we obtain a natural system of affine open subschemes of X, which we can endow with charts for the log structure. We will use these open subschemes in the definition of the associated blue scheme.

The following fact is a strengthening of [27, Lemma 1.6] and [53, Prop. 4.7] under this additional assumption on U.

Lemma 15.4. *Let $V = \operatorname{Spec}^K M_V$ an affine open of F_X. Assume that $U = \chi_X^{-1}(V)$ is affine. Then there is a chart $(M_V)_U \to \mathcal{M}_U$ such that the composition $M_V \to \mathcal{M}_U(U) \to \overline{\mathcal{M}}_U(U) = M_V$ is the identity on M_V.*

Proof. Since M_V is a fine monoid, it embeds into M_V^{gp}, which is a finitely generated abelian group. If $a^n = 1$ for an element $a \in M_V^{\mathrm{gp}}$ and $n \geqslant 1$, then we have $a \in M_V$ since M_V is saturated. Since M_V is sharp, we conclude that $a = 1$, which shows that M_V^{gp} is torsion free and hence a free abelian group of finite rank.

Therefore the multiplicative map $\mathcal{M}_U(U) \to \overline{\mathcal{M}}_U(U) = M_V$ admits a section $s : M_V \to \mathcal{M}_U(U)$. Since for an open subset $V' = \mathrm{Spec}^K M_{V'}$ of V, the restriction map $\rho : M_V \to M_{V'}$ is of the form $M_V \to S^{-1}M_V \to S^{-1}M_V/(S^{-1}M_V)^\times = M_{V'}$ for $S = \{h \in M_V | \rho(h) = 1\}$, the section s extends uniquely to a section $s_U : (M_V)_U \to \mathcal{M}_U$ of sheaves of monoids.

We are left with showing that s_U is a chart. Clearly, $s_U^a : (M_V)_U^a \to \mathcal{M}_U$ is a monomorphism of sheaves. It is surjective on stalks since $M_V = \overline{\mathcal{M}}_U(U) = \mathcal{M}_U(U)/\mathcal{O}_U(U)^\times$ and therefore $M_V \to \overline{\mathcal{M}}_{U,x}$ is a surjection for every $x \in U$. Thus s_U is a chart. □

For the rest of this section, we assume that the inverse image $U = \chi^{-1}(V)$ of any affine open V of F_X is affine.

Let $\iota : Y \to X$ be a closed immersion of k-schemes and $\alpha : \mathcal{M}_X \to \mathcal{O}_X$ the structure morphisms of sheaves of monoids on X. For every affine open $V = \mathrm{Spec}^K M_V$ of F_X and the inverse image $U = \chi_X^{-1}(V)$ and $W = \iota^{-1}(U)$, we obtain the morphism of sheaves of monoids

$$\iota^*\alpha^*\mathcal{M}_U \quad\longrightarrow\quad \iota^*\mathcal{O}_U \quad\longrightarrow\quad \mathcal{O}_W.$$

Taking global sections yields a multiplicative map $\eta_V : \mathcal{M}(U) \to \mathcal{O}_W(W) = S_V$. We define the blueprint $B_V = A_V /\!\!/ \mathcal{R}_V$ where

$$A_V \;=\; \eta_V\big(\mathcal{M}(U)\big) \cup \{0\} \qquad \text{and} \qquad \mathcal{R}_V \;=\; \Big\{ \sum a_i \equiv \sum b_j \,\Big|\, \sum a_i = \sum b_j \text{ in } S_V \Big\}.$$

Lemma 15.5. *An inclusion $V' \subset V$ of open affine Kato subfans of F_X induces a finite localization $B_V \to B_{V'}$.*

Proof. Let $V = \mathrm{Spec}^K M_V$ and $V' = \mathrm{Spec}^K M_{V'}$. Let $U = \chi_X^{-1}(V)$ and $U' = \chi_X^{-1}(V')$. Then the characteristic map χ_X yields the identifications $\overline{\mathcal{M}}_X(U) = M_V$ and $\overline{\mathcal{M}}_X(U') = M_{V'}$, and we obtain projections $\pi_V : \mathcal{M}_X(U) \to M_V$ and $\pi_{V'} : \mathcal{M}_X(U') \to M_{V'}$.

The inclusion $U' \subset U$ yields the restriction map $\rho : \mathcal{M}(U) \to \mathcal{M}(U')$ and the inclusion $V' \subset V$ yields the restriction map $\bar{\rho} : M_V \to M_{V'}$. Let $\bar{S}$ be $\bar{\rho}^{-1}(1)$ and $S = \pi_U^{-1}(\bar{S})$. Since $\bar{\rho} \circ \pi_U = \pi_{U'} \circ \rho$, the image $\rho(S)$ is contained in the fibre $\pi_{U'}^{-1}(1)$, which is the set $\mathcal{M}_X(U')^\times$ of invertible elements. Therefore ρ induces a multiplicative map $S^{-1}\mathcal{M}_X(U) \to \mathcal{M}_X(U')$.

We claim that this map is an isomorphism of monoids. By Lemma 15.4, there are sections $M_V \to \mathcal{M}_X(U)$ and $M_{V'} \to \mathcal{M}_X(U')$ to the projections π_V and $\pi_{V'}$, respectively, such that $S^{-1}\mathcal{M}_X(U) = S^{-1}\mathcal{O}_X(U)^\times M_V$ and $\mathcal{M}_X(U') = \mathcal{O}_X(U')^\times M_{V'}$. Thus the claim follows if we can show that $S_\alpha^{-1}\mathcal{O}_X(U) \to \mathcal{O}_X(U')$ is an isomorphism where $S_\alpha = \alpha(U)(S)$ is the image of S in $\mathcal{O}_X(U)$. This can be verified geometrically, i.e. it suffices to show that $\mathrm{Spec} S_\alpha^{-1}\mathcal{O}_X(U) \subset U'$. The former open subset of U consists of all points $x \in U$ such that $S \subset \mathcal{M}(U)$ is mapped to the units $\mathcal{M}_{X,x}^\times = \mathcal{O}_{X,x}^\times$ of the stalk of $\mathcal{M}_X$ in x. This means that S is sent to $\{1\}$ in $M_{V'} = \bar{S}^{-1}M_V/\bar{S}^{\mathrm{gp}}$. Therefore $x \in U'$, which shows that $S^{-1}\mathcal{M}_X(U) \to \mathcal{M}_X(U')$ is an isomorphism.

We conclude that the natural map $B_V \to B_{V'}$ induces an isomorphism $\eta_V(S)^{-1}A_V \to A_{V'}$ of monoids. It is an isomorphism $\eta_V(S)^{-1}B_V \to B_{V'}$ of blueprints since $R_{V'} = \eta(S)^{-1}R_V$ and consequently the preaddition of $B_{V'}$ is generated by the preaddition of B_V. This concludes the proof of the lemma. □

Since $\mathcal{M}$ is a logarithmic structure for X, the monoid $\mathcal{M}(U)$ contains $k^\times$. Therefore the monoid $\eta_V(\mathcal{M}(U)) \cup \{0\}$ contains k, i.e. A_V is a blue k-algebra, and $\mathrm{Spec} B_V$ is an affine blue k-scheme. By Lemma 15.5, the diagram $\mathcal{U}$ of all morphisms $\mathrm{Spec} B_{V'} \to \mathrm{Spec} B_V$ where $V' \subset V$

are open affine Kato subfans of F_X forms a commutative diagram $\mathcal{U}$ of affine blue k-schemes and open immersions.

We define the *blue k-scheme associated with $\alpha:\mathcal{M}_X\to\mathcal{O}_X$ and $\iota:Y\to X$* as $Z=\operatorname{colim}\mathcal{U}$. It comes with a morphism $\beta:Y\to Z$ of blue k-schemes and a morphism $\bar{\eta}:\iota^*\mathcal{M}_X\to\beta^*\mathcal{O}_Z$ of log structures for Y where $\beta^*\mathcal{O}_Z$ is the log structure associated with the prelog structure $\beta^\sharp:\beta^{-1}\mathcal{O}_Z\to\mathcal{O}_Y$. This means that the diagram

$$\begin{tikzcd} \iota^*\mathcal{M}_X \arrow[rr, "\bar{\eta}"] \arrow[dr, "\iota^*(\alpha)"'] & & \beta^*\mathcal{O}_Z \arrow[dl, "\beta^\sharp"] \\ & \mathcal{O}_Y & \end{tikzcd}$$

of morphisms of sheaves of monoids on Y commutes.

Theorem 15.6. *Let $\alpha:\mathcal{M}_X\to\mathcal{O}_X$ be a log structure for X with characteristic map $\chi:X\to F_X$. Assume that for every affine open V of F_X, the inverse image $U=\chi^{-1}(V)$ is affine. Let $\iota:Y\to X$ a closed immersion of k-schemes and Z be the associated blue k-scheme. Then the Ulirsch tropicalization* $\mathrm{Trop}^U_{\alpha,\iota}(Y)$ *is naturally homeomorphic to* $\mathrm{Bend}_v(Z)(\mathcal{O}_\mathbb{T})$ *and the diagram*

$$\begin{tikzcd} Y^\sqsupset \arrow[r, "\mathrm{trop}^U_{\alpha,\iota}(Y)"] \arrow[d, "\simeq"] & \mathrm{Trop}^U_{\alpha,\iota}(Y) \arrow[d, "\simeq"] \\ \mathrm{Bend}_v(Y)(\mathcal{O}_\mathbb{T}) \arrow[r, "\mathrm{Bend}_v(\beta)(\mathcal{O}_\mathbb{T})"] & \mathrm{Bend}_v(Z)(\mathcal{O}_\mathbb{T}) \end{tikzcd}$$

of continuous maps commutes.

Proof. Since all functors and maps in question are defined locally, we can assume that $X=\operatorname{Spec}R$ is an affine fine and saturated log scheme with affine Kato fan $F_X=\operatorname{Spec}^K M$. Then the closed immersion $\iota:Y\to X$ corresponds to a surjection $R\to S$ of rings. Let $\eta:\mathcal{M}_X(X)\to R\to S$ be the multiplicative map that is induced by the log structure $\alpha:\mathcal{M}_X\to\mathcal{O}_X$ and define $A=\eta(\mathcal{M}_X(X))\cup\{0\}$ and the preaddition $\mathcal{R}=\{\sum a_i\equiv\sum b_j\,|\,\sum a_i=\sum b_j \text{ in } S\}$ on A. Then the blue k-scheme associated with α and ι is $Z=\operatorname{Spec}B$ for the blueprint $B=A/\!\!/\mathcal{R}$.

Recall that the tropicalization of Y is defined as the image of $Y^\sqsupset$ under $\mathrm{trop}^U_\alpha:X^\sqsupset\to\mathrm{Trop}^U_\alpha(X)$ together the restriction of trop^U_α to $Y^\sqsupset$. Together with the identifications that we used to define the tropicalization map, we obtain the commutative diagram

$$\begin{tikzcd} & & & \mathrm{Val}_v(B,\mathcal{O}_\mathbb{T}) \arrow[d, dotted, "\Psi"] & & \\ Y^\sqsupset \arrow[r, phantom, "="] & \mathrm{Val}_v(S,\mathcal{O}_\mathbb{T}) \arrow[rru, "j^*"] \arrow[rr, "\mathrm{trop}^U_{\alpha,\iota}"] \arrow[d, hook] & & \mathrm{im}\big(\mathrm{trop}^U_\alpha(Y)\big) \arrow[r, phantom, "="] \arrow[d, hook] & \mathrm{Trop}^U(Y) \\ X^\sqsupset \arrow[r, phantom, "="] & \mathrm{Val}_v(R,\mathcal{O}_\mathbb{T}) \arrow[rr, "\mathrm{trop}^U_\alpha"] & & \mathrm{Hom}(M,\mathcal{O}_\mathbb{T}) \arrow[r, phantom, "="] & \mathrm{Trop}^U(X) \end{tikzcd}$$

where j^* is induced by the inclusion $j:B\to S$. In the following, we will define the dotted arrow Ψ in the diagram and show that it is a bijection that commutes with the other maps of the diagram.

Note that η restricts to a multiplicative map $\mathcal{M}_X(X)\to B$, which induces a multiplicative map $\bar{\eta}:M=\overline{\mathcal{M}}_X(X)\to B/B^\times$. Let $w:B\to\mathcal{O}_\mathbb{T}$ be a valuation that extends the trivial valuation $v:k\to\mathcal{O}_\mathbb{T}$. We define the multiplicative map $\Psi(w):M\to\mathcal{O}_\mathbb{T}$ as $a\mapsto w(a')$ where $a'\in B$ is a representative of $\bar{\eta}(a)\in B/B^\times$. Note that the definition $\Psi(w)$ is independent from the choice of a' since w sends $B^\times$ to $\mathcal{O}_T^\times=\{1\}$. This defines Ψ as a map from $\mathrm{Val}_v(B,\mathcal{O}_\mathbb{T})$ to $\mathrm{Hom}(M,\mathcal{O}_\mathbb{T})$. This map is injective since $\bar{\eta}(M)=B/B^\times-\{0\}$ and every valuation $w:B\to\mathcal{O}_\mathbb{T}$ maps 0 to 0. It commutes with trop^U_α and j^* since the latter maps are defined as the restrictions of valuations $w:R\to\mathcal{O}_\mathbb{T}$ to M and B, respectively.

The next step is to show that j^* is surjective. Since $\mathrm{trop}_\alpha^U : \mathrm{Val}_v(R, \mathcal{O}_\mathbb{T}) \to \mathrm{Hom}(M, \mathcal{O}_\mathbb{T})$ is surjective, every valuation $w : B \to \mathcal{O}_\mathbb{T}$ extends to a valuation $w' : R \to \mathcal{O}_\mathbb{T}$. Since the preaddition of B contains all relations of S, w' factors through S and defines a valuation $w'' : S \to \mathcal{O}_\mathbb{T}$. Thus $j^*(w'') = w$, which shows that j^* is surjective. As a consequence, $\Psi(w) = \mathrm{trop}_{\alpha,\iota}^U(w'')$ is contained in $\mathrm{Trop}_{\alpha,\iota}^U(Y)$ and $\Psi : \mathrm{Val}_v(B, \mathcal{O}_T) \to \mathrm{Trop}_{\alpha,\iota}^U(Y)$ is a bijection as claimed.

By Theorem 7.16, we have natural identifications $\mathrm{Val}_v(S, \mathcal{O}_\mathbb{T}) = \mathrm{Hom}_\mathbb{T}(\mathrm{Bend}_v(S), \mathcal{O}_\mathbb{T})$ and $\mathrm{Val}_v(B, \mathcal{O}_\mathbb{T}) = \mathrm{Hom}_\mathbb{T}(\mathrm{Bend}_v(B), \mathcal{O}_\mathbb{T})$. Under these identifications, j^* corresponds to $\mathrm{Bend}_v(j)^*$, which is equal to $\mathrm{Bend}_v(\beta)(\mathcal{O}_\mathbb{T})$. This establishes the commutativity of the diagram of the theorem.

It remains to show that the bijection $\mathrm{Trop}_{\alpha,\iota}^U(Y) \to \mathrm{Bend}_v(Z)(\mathcal{O}_\mathbb{T})$ is a homeomorphism. This can be shown by the same technique as used in the proof of Theorem 9.1, and we leave out the details. □

Remark 15.7. In some sense, the construction of the associated blue scheme is reverse to the association of a log scheme with a blue scheme, as considered in [33, section 7.3]. In the following section 15.7, we give this a precise meaning in terms of a universal property satisfied by the associated blue scheme.

Example 15.8 (Toric varieties)**.** Let Δ be a fan in $N_\mathbb{R}$ and $X(\Delta)$ be the associated toric k-variety. For a cone τ in Δ, let $M_\tau = \tau^\vee \cap N_\mathbb{Z}^\vee$ be the associated monoid, cf. section 9.1.

The associated log structure $\alpha : \mathcal{M}_X \to \mathcal{O}_X$ can be described as follows. For τ in Δ, we define $\mathcal{M}_{X_\tau} = (M_\tau)_{X_\tau}^a$ where the prelog-structure $(M_\tau)_{X_\tau} \to \mathcal{O}_{X_\tau}$ comes from the natural inclusion $M_\tau \to \mathbb{Z}[M_\tau]^+ = \mathcal{O}_X(X_\tau)$. Consequently, $\mathcal{M}_X$ comes with the charts $(M_\tau)_{X_\tau} \to \mathcal{M}_{X_\tau}$, and $(X, \mathcal{M}_X)$ is a fine and saturated log scheme.

The blue k-scheme Z associated with $X(\Delta)$ is defined locally as $Z_\tau = \mathrm{Spec}\, k[M_\tau]$, and it comes together with a canonical morphism $X(\Delta) \to Z$. By the very definition of the blue k-scheme Z' associated with $(X, \mathcal{M}_X)$, we see that Z and Z' are naturally isomorphic and that the canonical morphisms $\beta : X(\Delta) \to Z$ and $\beta' : X(\Delta) \to Z'$ agree.

Note that in this case, the closed embedding $\iota : Y \to X(\Delta)$ is the identity and that the morphism $\overline{\eta} : \iota^{-1}\mathcal{M}_X \to \beta^{-1}\mathcal{O}_Z$ of sheaves of monoids on X is an isomorphism.

For closed subvarieties $Y \hookrightarrow X(\Delta)$, we get two different blue models Y^{toric} and $Y^{\log}$ for Y. While Y^{toric} inherits its structure from the blue scheme structure of Z along the closed embedding $Y \hookrightarrow X(\Delta)$, and therefore is a closed blue subscheme of Z, the blue scheme $Y^{\log}$ is defined by the restriction of the boundary divisor of $X(\Delta)$ to Y. These two blue models of Y come with a morphism $Y^{\log} \to Y^{\mathrm{toric}}$, but this morphism is not an isomorphism in general.

An example where Y^{toric} and $Y^{\log}$ do not agree is the following. Consider the quadric Y defined by $x^2+y^2+z^2$ in $\mathbb{P}_k^{2,+} = \mathrm{Proj}\, k[x,y,z]^+$ with the canonical toric structure. Then Y^{toric} is the closed blue subscheme $\mathrm{Proj}\, k[x,y,z] /\!\!/ \langle x^2+y^2+z^2 \rangle$ of $\mathbb{P}_k^2$. The restriction D_Y of the boundary divisor D of $\mathbb{P}_k^{2,+}$ to Y consists of the six points of Y that lie on the intersection of Y with D. Therefore the Kato fan of the log structure on Y associated with D_Y has six closed points and does not embed into the Kato fan for the canonical log structure of $\mathbb{P}_k^{2,+}$, which has only three closed points. Consequently $Y^{\log}$ does not embed into $\mathbb{P}_k^2$.

Example 15.9 (Canonical log structure of a divisor)**.** Let X be an integral k-scheme of finite type and $H = H_1 + \cdots + H_r$ a Weil divisor where the H_i are pairwise coprime codimension one k-subschemes of X. Let $\alpha : \mathcal{M}_X \to \mathcal{O}_X$ be the canonical log structure of H. In the following, we give an explicit description of the associated blue k-scheme Z. Note that this generalizes the previous example.

Define $I = \{1,\dots,r\}$ and let U_J be the complement of $\bigcup_{i\in J} H_i$ in X for $J \subset I$. Note that $U_J \subset U_{J'}$ for $J' \subset J$, and in particular $U_I \subset U_J$ for all J. Define the blueprint $B_J = A_J /\!\!/ \mathcal{R}_J$ as

$$A_J \;=\; \mathcal{O}_X(U_J) \cap \mathcal{O}_X(U_I)^\times \quad \text{and} \quad \mathcal{R}_J \;=\; \Big\langle \sum a_i \equiv \sum b_j \;\Big|\; \sum a_i = \sum b_j \text{ in } \mathcal{O}_X(U_J) \Big\rangle.$$

We obtain induced morphisms $B_{J'} \to B_J$ for $J' \subset J$. Let $\mathcal{S}$ be the set of subsets of $J \subset I$ such that $B_J \to B_I$ is a finite localization. We denote by $\mathcal{V}$ the diagram of affine blue k-schemes $Z_J = \operatorname{Spec} B_J$ with $J \in \mathcal{S}$ together with the open immersions $\operatorname{Spec} B_J \to \operatorname{Spec} B_{J'}$ for $J' \subset J$. Then the blue k-scheme Z associated with the log structure α is the colimit of $\mathcal{V}$.

15.7. Recovering the Kato fan. The associated blue k-scheme Z together with $\beta: Y \to Z$ and $\overline{\eta}: \iota^*\mathcal{M}_X \to \beta^*\mathcal{O}_Z$ can be characterized as the universal blue k-scheme among all morphisms $\beta': Y \to Z'$ of blue k-schemes together with a morphism $\overline{\eta}': \iota^*\mathcal{M}_X \to \beta^*\mathcal{O}_{Z'}$ of log structures for Y. Therefore we do not rely on the existence of a Kato fan for the definition of the associated blue scheme. This allows us to extend the concept of tropicalization of log schemes to a wider class of log structures; cf. Example 15.11.

A posteriori, we can recover the Kato fan F_X and the extended cone complex $\overline{\Sigma}_X$ from the scheme theoretic tropicalization of a fine and saturated log scheme $Y = X$ over a field k as follows if we assume that $\chi^{-1}(U)$ is affine for all affine opens $U \subset F$ and if we assume that $\alpha: \mathcal{M}_X \to \mathcal{O}_X$ is a monomorphism, i.e. $\mathcal{M}_X(U) \to \mathcal{O}_X(U)$ is injective for all open subsets U of X. Let $\alpha: \mathcal{M}_X \to \mathcal{O}_X$ be the log structure of X and Z the associated blue k-scheme. Let $v: k \to \mathcal{O}_\mathbb{T}$ be the trivial valuation, which factors into the trivial valuation $v_0 : k \to \mathbb{B}$ followed by the unique inclusion $i: \mathbb{B} \to \mathcal{O}_\mathbb{T}$. Therefore, we have $\operatorname{Bend}_v(Z) = \operatorname{Bend}_{v_0}(Z) \otimes_\mathbb{B} \mathcal{O}_\mathbb{T}$. Conversely, any morphism $p: \mathcal{O}_\mathbb{T} \to \mathbb{B}$ induces the identification $\operatorname{Bend}_{v_0}(Z) = \operatorname{Bend}_v(Z) \otimes_{\mathcal{O}_\mathbb{T}} \mathbb{B}$ since $v: k \to \mathcal{O}_\mathbb{T}$ has image $\{0,1\}$.

Let $Z_0 = \operatorname{Bend}_{v_0}(Z)$. Since Z_0 is locally of finite type over the blue field $\mathbb{B}$, it has an associated subcanonical blue scheme $Z_0^{\mathrm{sub}} = \mathcal{F}(Z_0)$, as explained in Theorem 6.1. Due to the rigid topology of subcanonical blue schemes, cf. section 6.1, the functor $(-)^\bullet : \mathrm{OBlpr} \to \mathrm{Mon}$ extends to a functor from the category $\mathrm{Sch}^{\mathrm{can}}_{\mathbb{F}_1}$ of subcanonical blue schemes to the category $\mathrm{Sch}^{\mathrm{mon}}_{\mathbb{F}_1}$ of monoid schemes, in contrast to the negative result for geometric blue schemes. We denote the monoid scheme associated with Z_0^{can} by $Z_0^\bullet$.

Note that $Z_0^\bullet$ is integral. Therefore we can endow $Z_0^\bullet$ with the sheaf $\overline{\mathcal{O}}_{Z_0^\bullet}$ of strict monoids $\overline{\mathcal{O}}_{Z_0^\bullet}(V) = (\mathcal{O}_{Z_0}(V)^\bullet - \{0\}) / \mathcal{O}_{Z_0^\bullet}(V)^\times$ where $V \subset Z_0$ is open.

Theorem 15.10. *The monoidal space $(Z_0^\bullet, \overline{\mathcal{O}}_{Z_0^\bullet})$ is naturally isomorphic to the Kato fan F_X of X. Consequently, $\operatorname{Bend}_v(Z)(\mathcal{O}_\mathbb{T}) = F_X(\mathcal{O}_\mathbb{T})$ comes with the structure of an extended cone complex.*

Proof. It suffices to verify this theorem in the affine case. Thus let $X = \operatorname{Spec} R$, $Z = \operatorname{Spec} B$ and $F_X = \operatorname{Spec}^K M$.

Then $\operatorname{Bend}_{v_0}(B) = (B^\bullet \otimes_{k^\bullet} \mathbb{B}) /\!\!/ \operatorname{bend}_{v_0}(B)$. If the bend relation contains $a \equiv b$, then this relation is a sequence of generators of $\operatorname{bend}_{v_0}(B)$. In particular, $\operatorname{bend}_{v_0}(B)$ must contain a generator of the form $\sum c_j \equiv b$, which must be of the form $b + c \equiv b$ and come from $b \leqslant c$. Since B is algebraic, we have $b = c$ in B. This shows that the underlying monoid of $\operatorname{Bend}_{v_0}(B)$ is $B^\bullet \otimes_{k^\bullet} \mathbb{B}$.

Since v_0 is surjective with fibres $v_0^{-1}(0) = \{0\}$ and $v_0^{-1}(1) = k^\times$, we have $B^\bullet \otimes_{k^\bullet} \mathbb{B}^\bullet \simeq B^\bullet / k^\times$. It is easily verified that this implies that the association $\mathfrak{p} \mapsto \pi^{-1}(\mathfrak{p})$ defines a homeomorphism between the prime spectra of $\operatorname{Bend}_{v_0}(B)^\bullet = B^\bullet \otimes_{k^\bullet} \mathbb{B}^\bullet$ and $B^\bullet$.

The association $\mathfrak{p} \mapsto \mathfrak{p} \cup \{0\}$ defines a homeomorphism between the affine Kato fan of $\mathcal{M}_X(X)$ and the prime spectrum $B^\bullet = \mathcal{M}_X(X) \cup \{0\}$. Similar to the surjection π, the surjection $\mathcal{M}_X(X) \to M = \mathcal{M}_X(X) / \mathcal{M}_X(X)^\times$ induces a homeomorphism between the respective affine Kato fans.

The composition of these homeomorphisms yields a homeomorphism $(Z_0^\bullet, \overline{\mathcal{O}}_{Z_0^\bullet}) \to F_X$. We have

$$M \;=\; \mathcal{M}_X(X)/\mathcal{M}_X(X)^\times \;=\; (B^\bullet - \{0\})/B^\times \;=\; (\mathrm{Bend}_{v_0}(B)^\bullet - \{0\})/\mathrm{Bend}_{v_0}(B)^\times,$$

which shows that the monoids of global sections coincide. Since for both monoidal spaces $(Z_0^\bullet, \overline{\mathcal{O}}_{Z_0^\bullet})$ and F_X, local sections are defined in terms of localizations modulo units of M and $\mathrm{Bend}_{v_0}(B)^\bullet$, respectively, we conclude that $(Z_0^\bullet, \overline{\mathcal{O}}_{Z_0^\bullet})$ is naturally isomorphic to F_X as a monoidal space.

The second claim of the theorem follows since the structure of $\overline{\Sigma}_X = F_X(\mathcal{O}_\mathbb{T})$ as an extended cone complex is induced by the topology of $F_X = (Z_0^\bullet, \overline{\mathcal{O}}_{Z_0^\bullet})$. □

Example 15.11. We give some examples of blue schemes that are universal for log schemes which are not fine and saturated. As a first example, consider the log structure $\mathcal{M}_X = \mathcal{O}_X$, which is not coherent in general, and the trivial immersion $X \to X$. Then the associated blue k-scheme is X itself.

This can be generalized to the restriction of $\mathcal{O}_X$ to a proper closed subscheme, i.e. to $\mathcal{M}_X = \iota_*\mathcal{O}_Y$ where $\iota : Y \to X$ is a closed immersion. This log structure is not quasi-coherent and does not allow a strict morphism into a Kato fan in general. However, it easy to see that the associated blue k-scheme is Y.

The last example generalizes to restriction of fine and saturated log structures on X to closed subschemes Y. The associated blue k-scheme in this case is the very same as the one that we have defined in section 15.6.

15.8. What is new? The interpretation of the Ulirsch tropicalization as a scheme theoretic tropicalization enhances the topological spaces $\mathrm{Trop}^U_{\alpha,\iota}(Y)$ with a scheme structure, which was bound to subvarieties of toric varieties in terms of the Giansiracusa tropicalization so far. In particular, the scheme theoretic tropicalization endows the Ulirsch tropicalization intrinsically with a topology, which allows us to detach the Ulirsch tropicalization from its ambient extended cone complex.

A posteriori, we can recover the extended cone complex via the natural identification of the Kato fan with the prime spectrum of the underlying monoid scheme of the scheme theoretic tropicalization.

Besides these structural improvements of the Ulirsch tropicalization, we observe that the associated blue scheme Z of a fine and saturated log scheme X can be tropicalized along any valuation $v : k \to \mathbb{T}$. This yields an immediate answer to some questions posed in section 9.1 of the recent overview paper [2] by Abramovich, Chen, Marcus, Ulirsch and Wise.

References

[1] Dan Abramovich, Lucia Caporaso, and Sam Payne. The tropicalization of the moduli space of curves. *Ann. Sci. Éc. Norm. Supér.*, 48(4):765–809, 2015.

[2] Dan Abramovich, Qile Chen, Steffen Marcus, Martin Ulirsch, and Jonathan Wise. Skeletons and fans of logarithmic structures. Preprint, arXiv:1503.04343, 2015.

[3] Matt Baker. An introduction to Berkovich analytic spaces and non-archimedean potential theory on curves. In *p-adic geometry*, volume 45 of *Univ. Lecture Ser.*, pages 123–174, Amer. Math. Soc., Providence, RI, 2008.

[4] Matthew Baker, Sam Payne, and Joseph Rabinoff. Nonarchimedean geometry, tropicalization, and metrics on curves. To be published in *Algebraic Geometry*, arXiv:1104.0320, 2011.

[5] Matthew Baker, Sam Payne, and Joseph Rabinoff. On the structure of non-Archimedean analytic curves. In *Tropical and non-Archimedean geometry*, volume 605 of *Contemp. Math.*, pages 93–121. Amer. Math. Soc., Providence, RI, 2013.

[6] Banerjee, Soumya D. Tropical geometry over higher dimensional local fields. *J. Reine Angew. Math.*, 698:71–87, 2015.

[7] George M. Bergman. The logarithmic limit-set of an algebraic variety. *Trans. Amer. Math. Soc.*, 157:459–469, 1971.

[8] Vladimir G. Berkovich. *Spectral theory and analytic geometry over non-Archimedean fields*, volume 33 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1990.
[9] Vladimir G. Berkovich. Smooth p-adic analytic spaces are locally contractible. *Invent. Math.*, 137(1):1–84, 1999.
[10] Vladimir G. Berkovich. Analytic geometry over $\mathbb{F}_1$. Slides, 2011. Online available from `www.wisdom.weizmann.ac.il/~vova/Padova-slides_2011.pdf`.
[11] Robert Bieri and J. R. J. Groves. The geometry of the set of characters induced by valuations. *J. Reine Angew. Math.*, 347:168–195, 1984.
[12] Chenghao Chu, Oliver Lorscheid, and Rekha Santhanam. Sheaves and K-theory for $\mathbb{F}_1$-schemes. *Adv. Math.*, 229(4):2239–2286, 2012.
[13] Alain Connes and Caterina Consani. The hyperring of adèle classes. *J. Number Theory*, 131(2):159–194, 2011.
[14] Anton Deitmar. Schemes over $\mathbb{F}_1$. In *Number fields and function fields—two parallel worlds*, volume 239 of *Progr. Math.*, pages 87–100. Birkhäuser Boston, Boston, MA, 2005.
[15] Anton Deitmar. Congruence schemes. *Internat. J. Math.*, 24(2):1350009, 46, 2013.
[16] Nikolai Durov. New approach to Arakelov geometry. Thesis, `arXiv:0704.2030`, 2007.
[17] Manfred Einsiedler, Mikhail Kapranov, and Douglas Lind. Non-Archimedean amoebas and tropical varieties. *J. Reine Angew. Math.*, 601:139–157, 2006.
[18] Tyler Foster, Philipp Gross, and Sam Payne. Limits of tropicalizations. *Israel J. Math.*, 201(2):835–846, 2014.
[19] Tyler Foster and Dhruv Rangnathan. Hahn analytification and connectivity of higher rank tropical varieties. Preprint, arXiv:1504.07207, 2015.
[20] Kazuhiro Fujiwara and Fumiharu Kato. Foundations of rigid geometry I. Preprint, arXiv:1308.4734, 2013.
[21] Jeffrey Giansiracusa and Noah Giansiracusa. Equations of tropical varieties. Preprint, arXiv:1308.0042, 2013.
[22] Jeffrey Giansiracusa and Noah Giansiracusa. The universal tropicalization and the Berkovich analytification. Preprint, arXiv:1410.4348, 2014.
[23] Mark Gross and Bernd Siebert. Logarithmic Gromov-Witten invariants. *J. Amer. Math. Soc.*, 26(2):451–510, 2013.
[24] Walter Gubler, Joseph Rabinoff, and Annette Werner. Skeletons and tropicalizations. Preprint, arXiv:1404.7044, 2014.
[25] Walter Gubler, Joseph Rabinoff, and Annette Werner. Tropical skeletons. Preprint, arXiv:1508.01179, 2015.
[26] Takeshi Kajiwara. Tropical toric geometry. In *Toric topology*, volume 460 of *Contemp. Math.*, pages 197–207. Amer. Math. Soc., Providence, RI, 2008.
[27] Kazuya Kato. Toric singularities. *Amer. J. Math.*, 116(5):1073–1099, 1994.
[28] Marc Krasner. Approximation des corps valués complets de caractéristique $p \neq 0$ par ceux de caractéristique 0. In *Colloque d'algèbre supérieure, tenu à Bruxelles du 19 au 22 décembre 1956*, Centre Belge de Recherches Mathématiques, pages 129–206. Établissements Ceuterick, Louvain, 1957.
[29] D. R. LaTorre. On h-ideals and k-ideals in hemirings. *Publ. Math. Debrecen*, 12:219–226, 1965.
[30] Paul Lescot. Absolute algebra II—ideals and spectra. *J. Pure Appl. Algebra*, 215(7):1782–1790, 2011.
[31] Oliver Lorscheid. Blue schemes as relative schemes after Toën and Vaquié. Preprint, arXiv:1212.3261v3, 2012.
[32] Oliver Lorscheid. The geometry of blueprints. Part I: Algebraic background and scheme theory. *Adv. Math.*, 229(3):1804–1846, 2012.
[33] Oliver Lorscheid. A blueprinted view on $\mathbb{F}_1$-geometry. In *Absolute arithmetic and $\mathbb{F}_1$-geometry*, European Mathematical Society Publishing House, 2016.
[34] Oliver Lorscheid. Scheme theoretic tropicalization. First version of the present text, arXiv:1508.07949v1, 2015.
[35] Oliver Lorscheid and Cecília Salgado. Schemes as functors on topological rings. Preprint, arXiv:1410.1948, to appear in the Jouirnal of Number Theory, 2014.
[36] Diane Maclagan and Felipe Rincón. Tropical schemes, tropical cycles, and valuated matroids. Preprint, arXiv:1401.4654, 2014.
[37] Diane Maclagan and Bernd Sturmfels. *Introduction to tropical geometry*, volume 161 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2015.
[38] Andrew Macpherson. Skeleta in non-archimedean and tropical geometry. Preprint, arXiv:1311.0502, 2013.
[39] Florian Marty. Relative Zariski open objects. *J. K-Theory*, 10(1):9–39, 2012.
[40] Gregory Mikhalkin. Real algebraic curves, the moment map and amoebas. *Ann. of Math. (2)*, 151(1):309–326, 2000.
[41] Grigory Mikhalkin. Enumerative tropical algebraic geometry in $\mathbb{R}^2$. *J. Amer. Math. Soc.*, 18(2):313–377, 2005.
[42] Takeo Nishinou and Bernd Siebert. Toric degenerations of toric varieties and tropical curves. *Duke Math. J.*, 135(1):1–51, 2006.
[43] Dmitri O. Orlov. Quasicoherent sheaves in commutative and noncommutative geometry. *Izv. Ross. Akad. Nauk Ser. Mat.*, 67(3):119–138, 2003.

[44] Frédéric Paugam. Global analytic geometry. *J. Number Theory*, 129(10):2295–2327, 2009.
[45] Sam Payne. Analytification is the limit of all tropicalizations. *Math. Res. Lett.*, 16(3):543–556, 2009.
[46] Jérôme Poineau. La droite de Berkovich sur **Z**. *Astérisque*, 334, viii+xii+284 pp., 2010.
[47] Patrick Popescu-Pampu and Dmitry Stepanov. Local tropicalization. In *Algebraic and combinatorial aspects of tropical geometry*, volume 589 of *Contemp. Math.*, pages 253–316. Amer. Math. Soc., Providence, RI, 2013.
[48] David E. Speyer. Tropical geometry. Thesis. Online available at `www-personal.umich.edu/~speyer/thesis.pdf`, 2005.
[49] Dimitrios Stratigopoulos. Hyperanneaux non commutatifs: Hyperanneaux, hypercorps, hypermodules, hyperespaces vectoriels et leurs propriétés élémentaires. *C. R. Acad. Sci. Paris Sér. A-B*, 269:A489–A492, 1969.
[50] Amaury Thuillier. Géométrie toroïdale et géométrie analytique non archimédienne. Application au type d'homotopie de certains schémas formels. *Manuscripta Math.*, 123(4):381–451, 2007.
[51] Bertrand Toën and Michel Vaquié. Au-dessous de Spec $\mathbb{Z}$. *J. K-Theory*, 3(3):437–500, 2009.
[52] Ilya Tyomkin. Tropical geometry and correspondence theorems via toric stacks. *Math. Ann.*, 353(3):945–995, 2012.
[53] Martin Ulirsch. Functorial tropicalization of logarithmic schemes: the case of constant coefficients. Preprint, arXiv:1310.6269, 2013.
[54] Martin Ulirsch. Tropicalization is a non-Archimedean analytic stack quotient. Preprint, arXiv:1410.2216, 2014.
[55] Oleg Y. Viro. Hyperfields for Tropical Geometry I. Hyperfields and dequantization. Preprint, arXiv:1006.3034, 2010.
[56] Oleg Y. Viro. On basic concepts of tropical geometry. *Proc. Steklov Inst. Math.*, 273, No. 1:252-282, 2011.
[57] Jarosław Włodarczyk. *Embeddings in toric varieties and prevarieties*. J. Algebraic Geom. 2(4):705–726, 1003.

E-mail address: `oliver@impa.br`

INSTITUTO NACIONAL DE MATEMÁTICA PURA E APLICADA, ESTRADA DONA CASTORINA 110, RIO DE JANEIRO, BRAZIL